

Digital Signal Processing (DSP) (3 credits)

Signal is

function defines correspondence b/w 2 sets.

* Relation b/w discrete time unit impulse & unit step.

$$\delta[n] = u[n] - u[n-1]$$

* Defⁿ of DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

if seq. is of finite duration, then there exists a finite case of DTFT. This is known as DFT. ^{N length. that is} Limit 0 to N-1.

$X(e^{j\omega})$ are frequency samples at $\omega = \omega_k$

$$0 \leq k \leq N-1$$

* Defⁿ of DFT

$$X[k] = X(e^{j\omega}) \Big|_{\omega = \frac{2\pi k}{N}} \quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi nk}{N}}$$

* Twiddle factor

$$W_N = e^{-\frac{j2\pi}{N}}$$

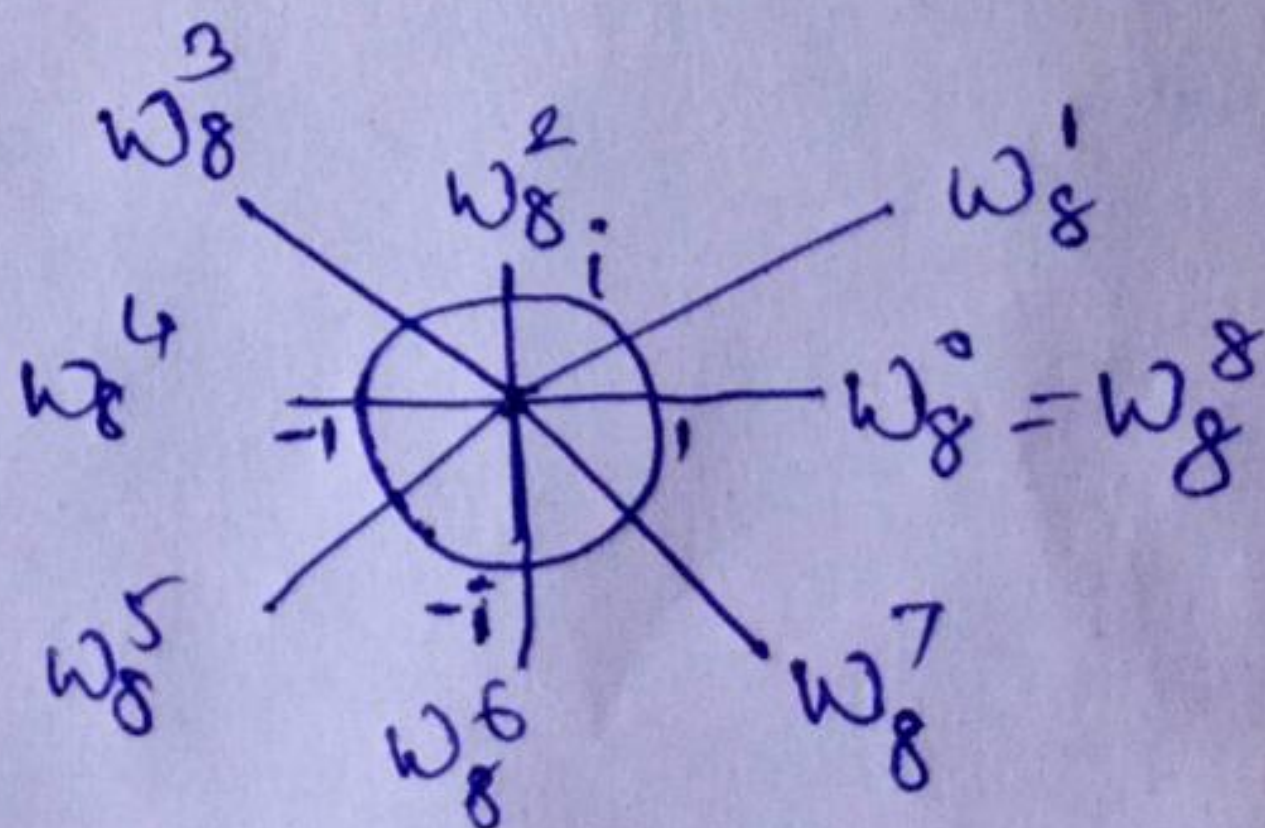
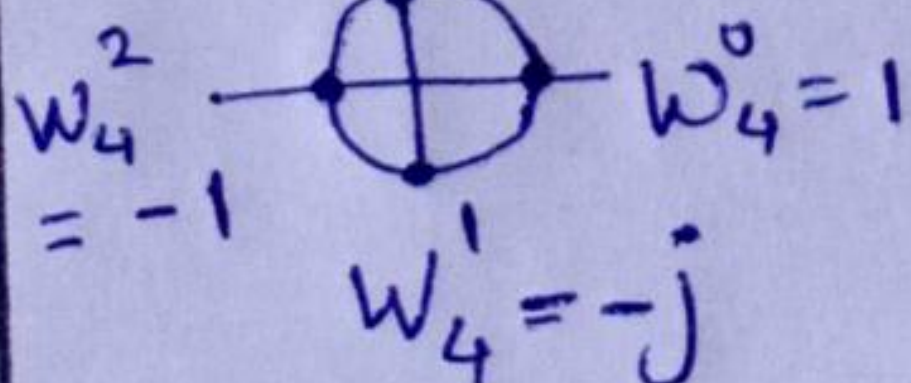
$$\therefore X[k] = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$\dots \dots \dots 0 \leq k \leq N-1$$

$$\text{IDFT} \rightarrow x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \dots \dots \dots 0 \leq n \leq N-1$$

* DFT as linear transformation

$W_4^3 = j \rightarrow$ 4-point DFT



$$W_8^0 = -W_8^4$$

$$W_8 = e^{2\pi j/8}$$

$$W^0 = 1 \quad X_N = W_N x_N$$

W_N = matrix of linear combination

$$x_n = \frac{1}{N} X_N W_N^* \rightarrow \text{complex conjugate}$$

- Frequency domain representation of discrete time sequence is given by DTFT.

- It expresses the sequence as weighted combination of complex exponential sequence $\{e^{j\omega}\}$

$\omega \rightarrow$ real normalized frequency variable.

Q.] Find N point DTFT of DT signal.

$$x(n) = a^n u(n)$$

$u(n) \rightarrow$ unit step signal.

DTFT $\rightarrow X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \frac{1}{1 - a e^{-j\omega}} ; |a e^{-j\omega}| < 1$$

DFT

$$X[k] = \frac{1}{1 - a e^{-j\omega}}$$

$$\omega = \frac{2\pi k}{N}$$

$$X[k] = \frac{1}{1 - a e^{-j\frac{2\pi k}{N}}}$$

$$\forall k = 0, 1, 2, \dots, N-1$$

★ Twiddle Factor Matrix

When $N = 4$
4 point DFT

$$\begin{bmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{bmatrix}$$

$$\begin{array}{c} W^3 W^7 j \\ | \\ W^2 W^6 \\ | \\ W^1 W^5 \\ | \\ -1 \end{array}$$

$$W_N^{nk} = e^{-j2\pi nk/N}$$

$$W_N^k = e^{-j2\pi k/N}$$

$$W_4^0 = \frac{e^{-j2\pi \cdot 0/4}}{1} = 1 \quad e^{-j2\pi \cdot 0/4} = 1$$

$$W_4^1 = e^{-j2\pi(1)/4} = e^{-j\pi/2} = \cos \frac{\pi}{2} - j \sin \frac{\pi}{2} = -j$$

$$W_4^2 = e^{-j2\pi 2/4} = e^{-j\pi} = \cos \pi - j \sin \pi = -1$$

$$W_4^3 = e^{-j2\pi \cdot 3/4} = e^{-j3\pi/2} = e^{j\pi/2} = \cos \frac{\pi}{2} + j \sin \frac{\pi}{2} = j$$

$$\therefore W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Q.] $x(n) = \{1, 0, 0, 1\}$ Find DFT

$$\rightarrow x_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \therefore X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

// for 3 samples eg $\{1, 0, 1\}$

$x(n)$ will be $\{1, 0, 1, 0\}$

2 samples $\{1, 1\}$

$x(n) = \{1, 1, 0, 0\}$

$$X_4 = \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}$$

$$X(k) = \{2, 1+j, 0, 1-j\}$$

Q.] Find DFT using matrix method

$$x(n) = \{0, 1, 2, 3\}$$

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \{6, 2j-2, -2, -2j-2\}$$

$$\text{IDFT} = X_N = \frac{1}{N} W_N^* X_N \quad \xrightarrow{\text{complex conjugate.}}$$

given that $N-1 = 3 \quad \therefore N = 4$

$$W_4^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}$$

$$X_N = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}$$

$$= \frac{1}{4} [2 + 1 + j + 0 + 1 - j] = \frac{1}{4} [4] = 1$$

$$\textcircled{1} = \frac{1}{4} [4] = \underline{1}$$

$$\begin{aligned} \textcircled{2} \quad & \frac{1}{4} \{ 2 + j(1+j) - j(1-j) \} \\ & \frac{1}{4} \{ 2 + \cancel{j} + j^2 - \cancel{j} + j^2 \} \\ & \frac{1}{4} \{ 2 + 2j^2 \} = \underline{\underline{0}} \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad & \frac{1}{4} \{ 2 - (1+j) - (1-j) \} \\ & = \underline{\underline{0}} \end{aligned}$$

$$\begin{aligned} \textcircled{4} \quad & \frac{1}{4} \{ 2 - j(1+j) + j(1-j) \} \\ & = \underline{\underline{1}} \end{aligned}$$

$$\therefore \underline{\underline{1001}} = \text{IDFT}$$

$$\text{IDFT} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 6 \\ 2j-2 \\ -2 \\ -2j-2 \end{bmatrix}$$

$$\frac{1}{4} \{ 6 + \cancel{2j} - 2 - 2 - \cancel{2j} - 2 \} = \underline{\underline{0}}$$

$$\begin{aligned} & \frac{1}{4} \{ 6 + j(2j-2) - (-2) - j(-2j-2) \} \\ = & \frac{1}{4} \{ 6 + 2j^2 - \cancel{2j} + 2 + 2j^2 + \cancel{2j} \} \\ & 8 + 4(-1) = \underline{\underline{1}} \end{aligned}$$

$$\frac{1}{4} \{ 6 - \cancel{2j} + 2 - 2 + \cancel{2j} + 2 \} = \underline{\underline{2}}$$

$$\frac{1}{4} \{ 6 - 2j^2 + \cancel{2j} + 2 - 2j^2 - \cancel{2j} \} = \underline{\underline{3}}$$

0 1 2 3

★ Relationship b/w z transform & DFT

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \rightarrow \text{z transform of a Seq. } x(n)$$

$$z_k = e^{j \frac{2\pi k}{N}} \quad \forall k=0, 1, 2, \dots$$

$$X(k) = X(z) \Big|_{z=e^{j \frac{2\pi k}{N}}}$$

$$\therefore X(k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j \frac{2\pi n k}{N}} \dots \dots \dots \textcircled{A}$$

$$\omega_k = 2\pi k/N \quad \text{where } k=0, 1, 2, \dots, N-1$$

If we consider sequence $x(n)$ has finite duration we can recover the original seq. by taking IDFT

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \quad \therefore X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j \frac{2\pi n k}{N}} \right] z^{-n}$$

$$\therefore X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j \frac{2\pi n k}{N}} \right] * z^{-n}$$

$x(n)$ represented in $X(k)$ IDFT form.

$$\therefore X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j \frac{2\pi k}{N}} z^{-1} \right)^n$$

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) * \frac{1 - \left[e^{j \frac{2\pi k}{N}} z^{-1} \right]^N}{1 - e^{j \frac{2\pi k}{N}} z^{-1}}$$

Ref.

$$\sum_{n=0}^{N-1} a^n = N \quad \text{---} \quad \forall a=1$$

$$= \frac{1-a^N}{1-a} \quad \forall a \neq 1$$

Here $(e^{j2\pi k/N} z^{-1})^N = e^{j2\pi kN/N} \cdot z^{-N} = (1) \cdot z^{-N} = z^{-N}$

$$X(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{x(k)}{1-e^{j2\pi k/N} z^{-1}}$$

when evaluated on unit circle.

This is the relationship b/w z transform & DFT.

$$\begin{aligned} X(z) &= \frac{1}{N} \sum_{k=0}^{N-1} x(k) \cdot \frac{1-z^{-N}}{1-e^{j2\pi k/N} z^{-1}} \\ &= \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{x(k)}{1-e^{j2\pi k/N} z^{-1}} \end{aligned}$$

Periodicity:

If discrete time signal is periodic then the N point DFT of the sequence is also periodic.

If $x(k)$ is N point DFT of $x(n)$

$x(n) \xleftrightarrow{\text{DFT}} x[k]$ then this prop. is stated as

$$x(n+N) = x(n) \quad \text{for all values of } n$$

$$X(k+N) = X(k) \quad \text{for all values of } k.$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

Replace k by $k+N$

$$\begin{aligned} X(k+N) &= \sum_{n=0}^{N-1} x(n) W_N^{n(k+N)} \\ &= \sum_{n=0}^{N-1} x(n) W_N^{nk} \cdot W_N^{nN} \quad \text{--- (A)} \end{aligned}$$

We know that W_N is twiddle factor.

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\therefore W_N^{nN} = \left[e^{-j\frac{2\pi}{N}} \right]^{nN} = e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n)$$

$$\cancel{W_N^{nk}} = \left[e^{-j\frac{2\pi}{N}} \right]^{nk} = e^{-\frac{j2\pi nk}{N}}$$

$$\cos(2\pi n) = 1, \quad \sin 2\pi n = 0$$

$$\therefore W_N^{nN} = 1$$

$$\therefore X(k+N) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nk}$$

$$\therefore X(k+N) = X(k)$$

Hence proved.

Linearity property: It states that DFT of linear combinatⁿ of 2 or more signals is equal to sum of DFT of individual sequences.

$$\text{If } x_1(n) \xrightarrow[N]{\text{DFT}} X_1[k]$$

$$x_2(n) \xrightarrow[N]{\text{DFT}} X_2[k]$$

$$ax_1(n) + bx_2(n) \xrightarrow[N]{\text{DFT}} aX_1[k] + bX_2[k]$$

a & b these are constants.

Acc. to defⁿ of DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$x(n) = x_1(n) + x_2(n)$$

$$X[k] = \sum_{n=0}^{N-1} [a x_1(n) + b x_2(n)] W_N^{nk}$$

$$= \sum_{n=0}^{N-1} a x_1(n) W_N^{nk} + \sum_{n=0}^{N-1} b x_2(n) W_N^{nk}$$

$$= a \sum_{n=0}^{N-1} x_1(n) W_N^{nk} + b \sum_{n=0}^{N-1} x_2(n) W_N^{nk}$$

$$X[k] = a X_1[k] + b X_2[k]$$

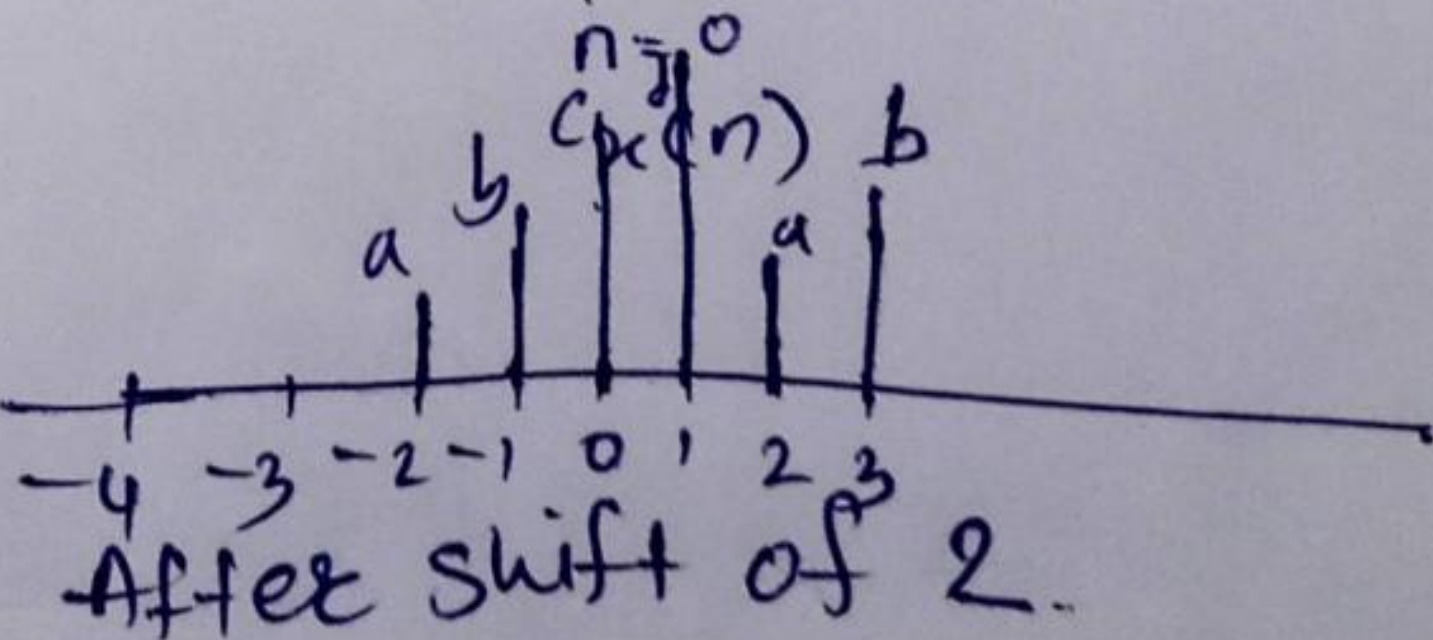
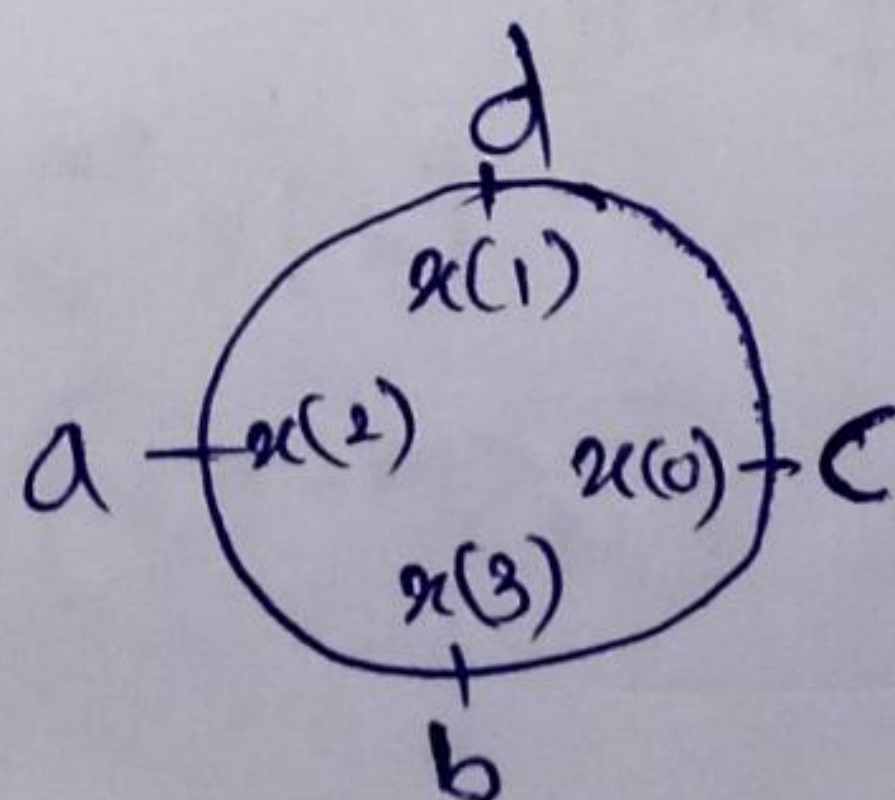
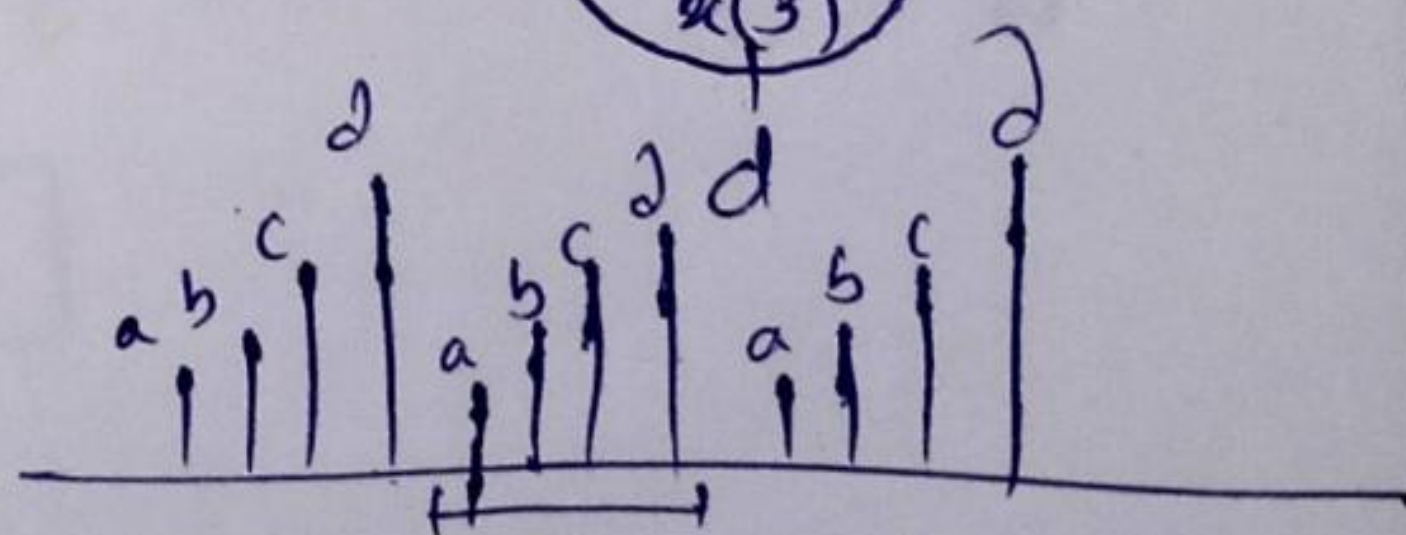
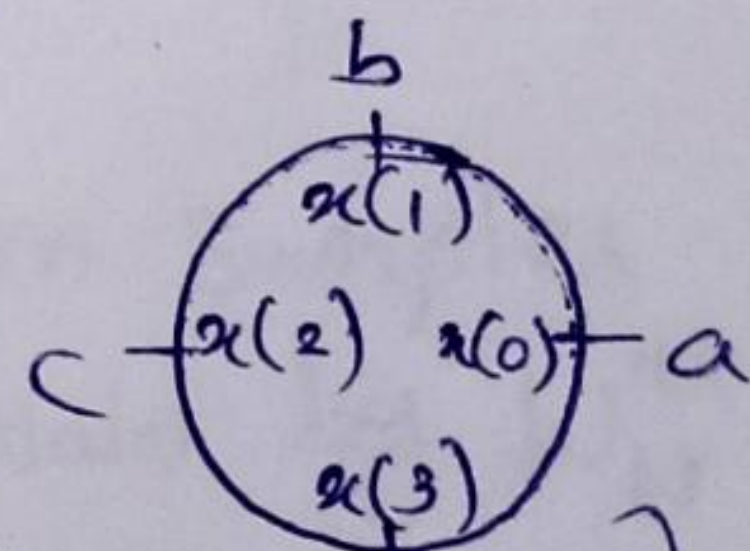
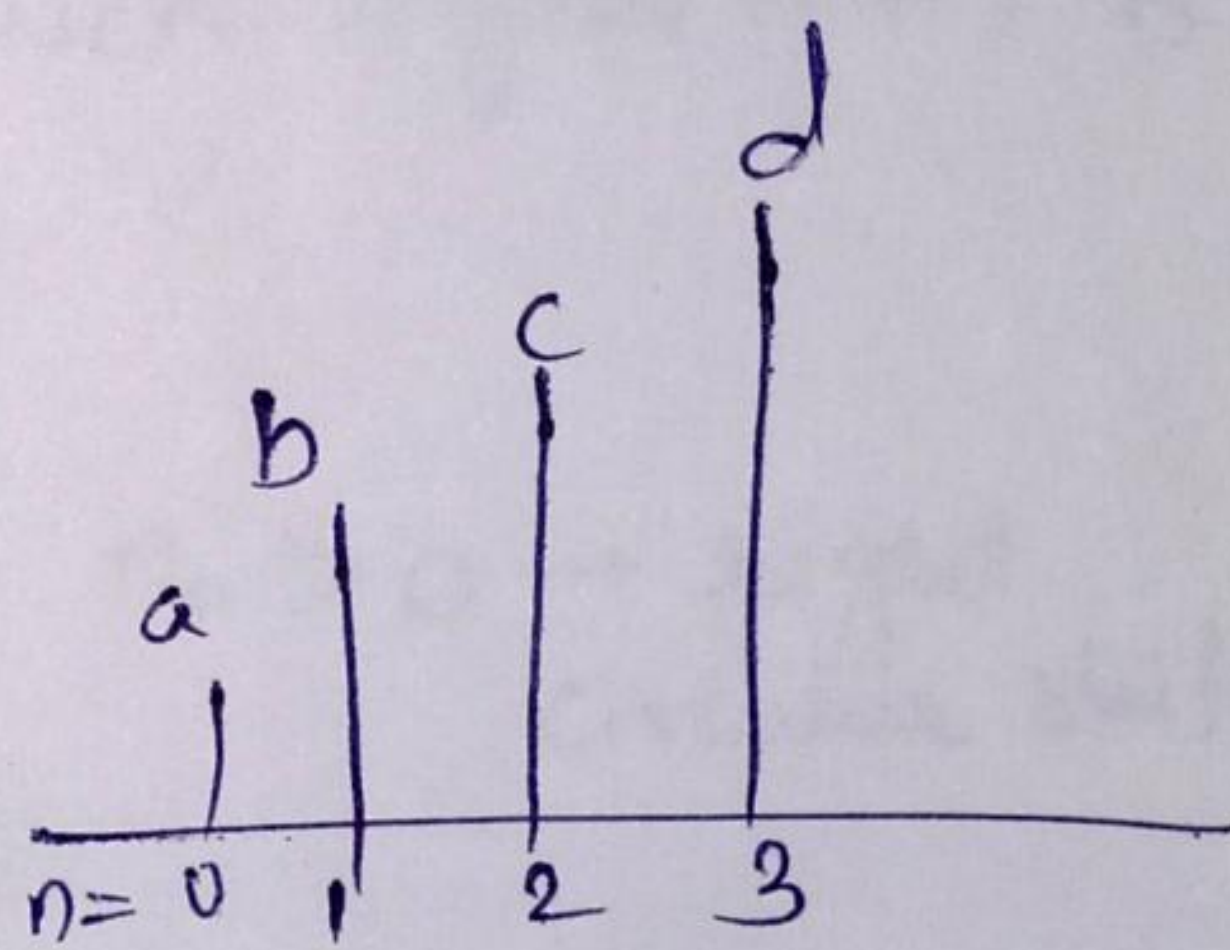
Hence proved.

Circular shift of a sequence.

$x(n) \rightarrow$ periodic seq.

$x_p(n)$ with period of N

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$



$$x(n) = \{1, 2, 3, 4\} \quad N=4$$

$$x_p(n) = \{\dots 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4 \dots\}$$

$$x_p(n-2) = \{3, 4, 1, 2\}$$

This relation of circular shift is given as

$$x^a(n) = x(n-n_0, \text{ modulo } N)$$

for $x_p(n-2)_4$ where $n_0=2, N=4$

divide $(n-n_0)$ by N & retain the remainder only.

$$x^a(n) = x \langle n-n_0 \rangle_N$$

n_0 = No. of samples by which signal $x(n)$ is delayed.

$N \Rightarrow$ point DFT

$$x^a(n) = x_p(n-2)_4 \quad \text{if } n_0 > 0 \rightarrow \text{right circular shift}$$

if $n_0 \leq 0 \rightarrow$ left circular shift.

* Circular time shift: It states that "shifting a seq. in time domain by m samples is equi. to multiplying a seq. in freq. domain by W_N^{mk} "

$$\underset{\text{Time}}{g \langle n-m \rangle_N} \xleftrightarrow[N]{\text{DFT}} \underset{\text{freq.}}{W_N^{km} G[k]}$$

$$\text{If } G[k] = W_N^{mk} X[k]$$

$$g(n) = x \langle n-m \rangle_N$$

Begin with inverse DFT

$$g[n] = \frac{1}{N} \sum_{k=0}^{N-1} G[k] W_N^{-nk}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{mk} W_N^{-nk}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-k(n-m)}$$

$$= x[n-m] = x[\langle n-m \rangle_N]$$

* Circular shift in frequency domain (Quadrature modulation theorem).

The inverse DFT of the circular frequency shifted

DFT $G[k] = X[\langle k-m \rangle_N]$ with m an integer is given

as $g[n] = W_N^{-mn} x[n]$

Proof: If $g[n] = W_N^{-mn} x[n]$ then

$$G[k] = X[\langle k-m \rangle_N]$$

Begin with DFT

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{nk}$$

$$= \sum_{n=0}^{N-1} W_N^{-mn} x[n] W_N^{nk}$$

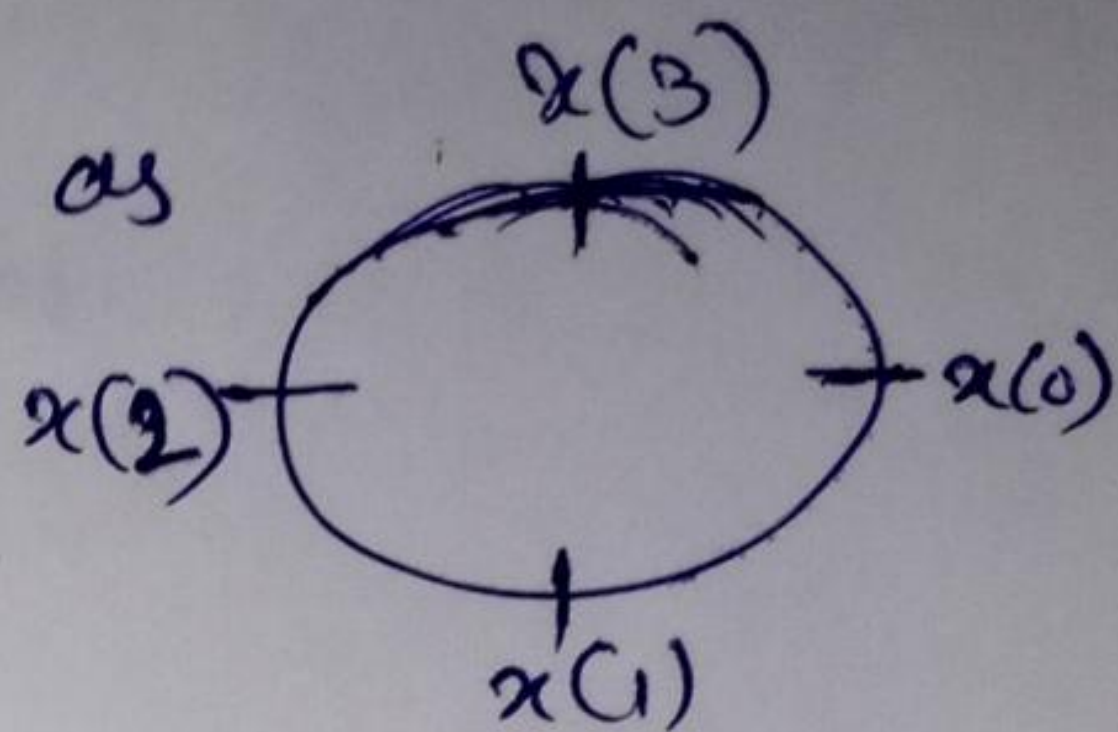
$$= \sum_{n=0}^{N-1} x[n] W_N^{n(k-m)}$$

$$= X[\langle k-m \rangle_N]$$

$$W_N^{-mn} x[n] \xrightarrow[N]{\text{DFT}} X[\langle k-m \rangle_N]$$

* Circularly folded sequence is defined as

$x(-n)$ plotted in a clockwise direction



* Circularly even sequence:

if seq. satisfies that $x(n) = x(N-n) \quad \forall \quad 1 \leq n \leq N-1$

$$x(n) = \{x(0), x(1), x(2), x(3)\} \quad N=4$$

$$1 \leq n \leq N-1$$

$$n=1 \quad x(1) = x(4-1)$$

$$x(1) = x(3)$$

$$n=2 \quad x(2) = x(2)$$

$$n=3 \quad x(3) = x(1)$$

$$n=1, 2, 3$$

* Circularly odd sequence:

If $x(n) = -x(N-n) \quad \forall \quad 1 \leq n \leq N-1$ then it is circularly odd seq.

$$x(n) = \{2, -3, 0, 3\} \quad N=4$$

$$n=1 \quad x(3) = -x(1)$$

$$n=1, 2, 3$$

$$n=2 \quad x(4-2) = -x(2)$$

$$x(2) = -x(2)$$

$$n=3 \quad x(1) = -x(3)$$

* To reduce no. of complex calculations & speed of operation, algorithms were developed by Cooley & Tukey.

↓ fast
FFT (Fourier ~~fast~~ transform)

highly efficient procedure for DFT for finite terms

calculation of DFT coefficients is calculated using this procedure iteratively.

Application → 1) Digital Spectrum analysis.

2) filter simulation.

3) Auto co-relation

4) Pattern recognition.

Reduction in computation time, Improvement in performance → Advantages of 2 methods →

DIT & DIF (Decimation in time & frequency)

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \quad 0 \leq K \leq N-1$$

$$W_N = e^{-j2\pi/N}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad 0 \leq K \leq N-1$$

$$= \sum_{n=0}^{N-1} \{ \text{Re}[x(n)] + j \text{Im}[x(n)] \}$$

$$\{ \text{Re}[W_N^{nk}] + j \text{Im}[W_N^{nk}] \}$$

* We require complex additions $\Rightarrow N-1$ for 1 value of k .
for all values of $k \Rightarrow N * N-1$

* for 1 value of k no. of multiplications $\Rightarrow N$

& for all values $\Rightarrow N^2$

DIT (Decimation in time algorithm)

(Radix 2 FFT algorithm) → Divide entire seq. in 2 sub parts.

$$x(n) \rightarrow N \quad X(k) \rightarrow N.$$

$$\therefore N = 2^V$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j \frac{2\pi nk}{N}} \quad \forall k=0, 1, \dots, N-1$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad \forall k=0, 1, 2, \dots, N-1$$

We will divide i/p seq. $x(n)$ in even indexed & odd indexed seq. samples.

$$X(k) = \underbrace{\sum_{n=0}^{N-1} x(n) W_N^{nk}}_{\text{even}} + \underbrace{\sum_{n=0}^{N-1} x(n) W_N^{nk}}_{\text{odd.}}$$

$$x_e[n] = x(2m) \text{ even samples.}$$

$$x_o[n] = x(2m+1) \text{ odd samples.}$$

$$X[k] = \sum_{m=0}^{N/2-1} x(2m) W_N^{2mk} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{(2m+1)k}$$

$$X[k] = \sum_{m=0}^{N/2-1} x(2m) W_N^{2mk} + W_N^k \sum_{m=0}^{N/2-1} x(2m+1) W_N^{2mk}$$

$$W_N^{2mk} = e^{-j \frac{2\pi 2mk}{N}} = e^{-j \frac{2\pi mk}{N/2}}$$

$$W_N^{2mk} = W_{N/2}^{mk}$$

$$W_N^{2nk} = W_{N/2}^{nk}$$

$$W_N^{(2m+1)k} = e^{-j \frac{2\pi (2m+1)k}{N}} = e^{-j \frac{2\pi mk}{N/2}} \cdot e^{-j \frac{2\pi k}{N}}$$

$$= W_{N/2}^{mk} \cdot W_N^k = W_{N/2}^{nk} \cdot W_N^k$$

$$x(k) = \sum_{n=0}^{N/2-1} x_e[n] \cdot W_{N/2}^{nk} + \sum_{n=0}^{N/2-1} x_o[n] W_{N/2}^{nk}$$

$$\boxed{X(k) = X_e(k) + W_N^k X_o(k)} \quad \forall k=0, 1, 2, \dots, (N/2-1) \quad \text{--- (A)}$$

Each of the sum in this eqⁿ is $N/2$ point DFT. Although the k index ranges from 0 to $N-1$, each of the sum is calculated only for $k=0, 1, \dots, (N/2-1)$. Since they are periodic in k with period $N/2$, twiddle factor is also periodic with period $N/2$.

$$\therefore W_N^k = W_N^{k+N/2} = W_N^k \cdot W_N^{N/2}$$

$$W_N^{N/2} = e^{-j \frac{2\pi N/2}{N}} = e^{-j\pi} = -1$$

$$\boxed{W_N^{k+N/2} = -W_N^k}$$

$$X(k+N/2) = X_e(k) - W_N^k X_o(k)$$

$$\forall k=0, 1, 2, \dots, (N/2-1)$$

$$\boxed{X(k) = X_e(k-N/2) - W_N^{k-N/2} X_o(k-N/2)} \quad \text{--- (B)}$$

$$\forall k=N/2, (N/2+1), \dots, (N-1)$$

Required multiplications for $N/2$ point DFT are $(N/2)^2$ for even samples & $(N/2)^2$ for odd samples.

$$\therefore \text{total multiplications} = 2 \left(\frac{N}{2} \right)^2$$

Required complex additions to combine $N/2$ point DFTs

$$\text{are } \frac{N}{2}(N/2-1) + \frac{N}{2}(N/2-1) + N$$

$$= \frac{N^2}{2}$$

consider $N=8$, with direct method calculate how many no. of multiplications $\times$ required.

$$= N^2 = \underline{\underline{64}}$$

* Total no. of multiplications when we divide total 8eq. in 2 parts.

$$X(k) = X_e(k) + W_N^k \cdot X_o(k)$$

$\downarrow$
 $(\frac{N}{2})^2$

$\downarrow$
 N

$\downarrow$
 $(\frac{N}{2})^2$

$$\therefore \text{total } (\frac{N}{2})^2 + (\frac{N}{2})^2 + N = \frac{N^2}{2} + N$$

$$\therefore \text{for } N=8 \quad \text{total multiplications} = \underline{\underline{40}}$$

$\therefore$ DIT is more efficient.

Let $N=8$

$N/2$	$X_e(k)$	$X_o(k)$
	4	4

$$X_e(0) = X(0)$$

$$X_e(1) = X(2)$$

$$X_e(2) = X(4)$$

$$X_e(3) = X(6)$$

$$X_o(0) = X(1)$$

$$X_o(1) = X(3)$$

$$X_o(2) = X(5)$$

$$X_o(3) = X(7)$$

$$X(k) = X_e(k) + W_N^k X_o(k) \quad \text{--- } 0 \leq k \leq 3$$

$$X(k) = X_e(k-4) + W_8^{k-4} X_o(k-4) \quad \text{--- } 4 \leq k \leq 8$$

By substituting various values of k we will get

$$X(0) = X_e(0) + W_8^0 X_o(0)$$

$$X(1) = X_e(1) + W_8^1 X_o(1)$$

$$X(2) = X_e(2) + W_8^2 X_o(2)$$

$$X(3) = X_e(3) + W_8^3 X_o(3)$$

$$x(4) = x_e(0) - w_8^0 x_o(0)$$

$$x(5) = x_e(1) - w_8^1 x_o(1)$$

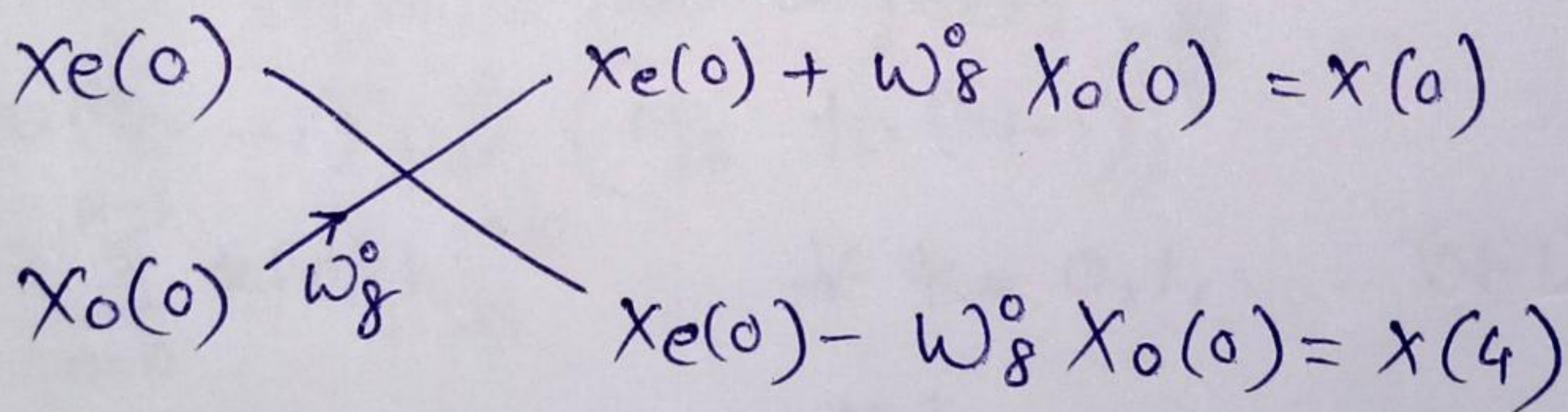
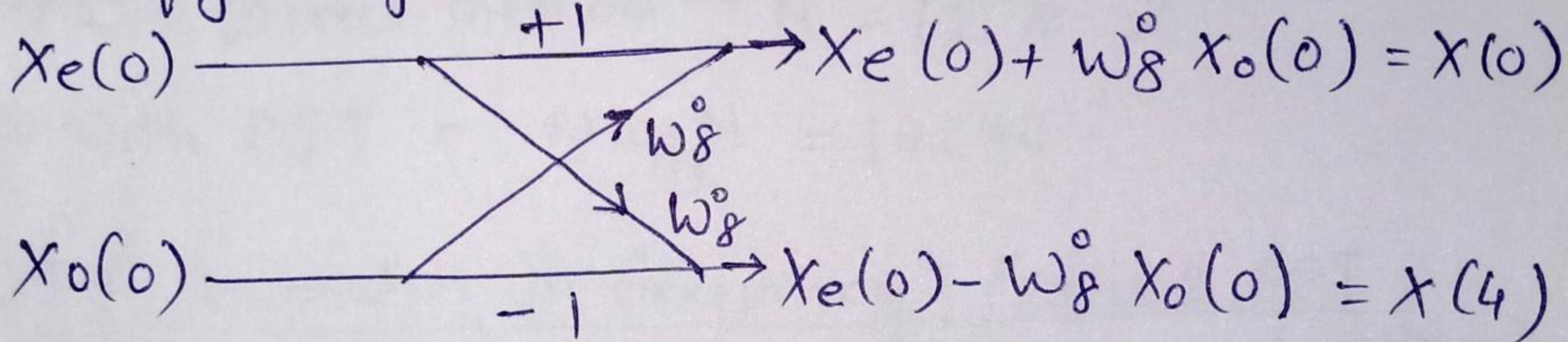
$$x(6) = x_e(2) - w_8^2 x_o(2)$$

$$x(7) = x_e(3) - w_8^3 x_o(3)$$

$\therefore x(0)$ & $x(4)$ have same I/P

Similarly $x(1)$ & $x(5)$

* Butterfly diagram



$$N/2 \text{ point DFT} \rightarrow 2\left(\frac{N}{2}\right)^2 + N = \frac{N^2}{2} + N$$

$$N/4 \text{ point DFT} \rightarrow 2\left[2\left(\frac{N}{4}\right)^2 + \frac{N}{2}\right] + N = \frac{N^2}{4} + 2N$$

$$N/8 \text{ point DFT} \rightarrow 2\left[2\left[2\left(\frac{N}{8}\right)^2 + \frac{N}{4}\right] + \frac{N}{2}\right] + N = \frac{N^2}{8} + 3N$$

$$\frac{N^2}{2^p} = 1 = \frac{N^2}{2^p} + pN$$

$$8 = 2^3$$

$$= \frac{N^2}{N} + N \log_2 N$$

$$= N + N \log_2 N$$

If N is power of 2 we can recursively break the DFT into $\log_2 N$

for $N=2^{10}$ Find ^{no. of} complex calculations

1> With Direct method $\rightarrow N^2 = (2^{10})^2 = 2^{20}$

2> With DIT $= N \log_2 N = 10240$

* DIF (Decimation in frequency) (Radix 2 FFT algorithm)

Divide the sequence into 2 parts, ~~1~~
 $(0 \text{ to } (N/2 - 1))$ & $(N/2 \text{ to } (N-1))$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad \forall k = 0, 1, \dots, (N-1).$$

$$X(k) = \underbrace{\sum_{n=0}^{N/2-1} x(n) W_N^{nk}}_{\text{Term A}} + \underbrace{\sum_{n=N/2}^{N-1} x(n) W_N^{nk}}_{\text{Term B}}$$

Consider Term B.

$$= \sum_{n=N/2}^{N-1} x(n) W_N^{nk}$$

to change the limits from 0 to $N/2 - 1$

Let $p = n - N/2$, $n = p + N/2$

Substituting these values in Term B.

$$\text{Term B} = \sum_{p=0}^{N/2-1} x[p + N/2] W_N^{(p+N/2)k}$$

$$= \sum_{n=0}^{N/2-1} x[p + N/2] W_N^{pk} \cdot W_N^{N/2 k}$$

$$W_N^{N/2 k} = e^{-j \frac{2\pi}{N} \frac{N}{2} k} = e^{-j \pi k} = (-1)^k \dots \forall k=0,1,$$

$$\therefore \text{Term B} = \sum_{p=0}^{N/2-1} x[p + N/2] W_N^{pk} (-1)^k$$

Substitute $p=n$

$$\begin{aligned} \text{Term B} &= \sum_{n=0}^{N/2-1} x[n + N/2] W_N^{nk} (-1)^k \\ &= (-1)^k \sum_{n=0}^{N/2-1} x[n + N/2] W_N^{nk} \end{aligned}$$

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + (-1)^k \sum_{n=0}^{N/2-1} x(n + N/2) W_N^{nk}$$

$$\therefore X(k) = \sum_{n=0}^{N/2-1} \left\{ x(n) + (-1)^k x(n + N/2) \right\} W_N^{nk}$$

Now, split $x(k)$ into even & odd numbered samples

Substitute $k=2k$ (even numbered samples)

$$X(2k) = \sum_{n=0}^{N/2-1} \left\{ x(n) + (-1)^{2k} x(n + N/2) \right\} W_N^{2nk}$$

$$W_N^{2nk} = W_{N/2}^{nk} \quad \& \quad (-1)^{2k} = 1$$

$$\boxed{X(2k) = \sum_{n=0}^{N/2-1} \left\{ x(n) + x(n + N/2) \right\} W_{N/2}^{nk}} \quad \text{--- (a)}$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} \left\{ x(n) + (-1)^{2k+1} x(n+N/2) \right\} W_N^{n(2k+1)}$$

Since $(-1)^{2k+1} = -1$ & $W_N^{2nk} = W_{N/2}^{nk}$

$$X(2k+1) = \sum_{n=0}^{N/2-1} \left\{ x(n) - x(n+N/2) \right\} W_N^{n(2k+1)}$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} \left\{ x(n) - x(n+N/2) \right\} W_N^{2nk} \cdot W_N^n$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} \left\{ x[n] - x[n+N/2] \right\} W_{N/2}^{nk} W_N^n$$

Circular Convolution: Consider two length N sequences $g[m]$ & $h[n]$ defined for $0 \leq n \leq N-1$.

Linear convolution: It will result into sequence with length = $(2N-1)$

$$y[n] = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

For N samples, linear convolution is represented as

$$y_L[n] = \sum_{m=0}^{N-1} g[m] h[n-m] \quad \forall 0 \leq n \leq 2N-2$$

Here both N length sequences are ^{zero} padded to extend their length upto $2N-1$

The first non zero value of

$$y_L[n] \Rightarrow y_L[0] = g[0]h[0]$$

$$y_L[n] = g[N-1]h[N-1]$$

1) Time reversal of $h[N]$ 2) linear shifting.
2 steps involved in linear convolution.

To get o/p samples, same as no. of i/p samples, then we should go for circular convolution.

To develop a convolution type operation that result in length N , we first apply circular time reversal & then circular shift. The Result obtained is called as circular convolution.

It is represented as

$$y_c(n) = \sum_{m=0}^{N-1} g[m] h[\langle n-m \rangle_N]$$

This is N point circular convolution.

$$y_c(n) = g(n) \textcircled{N} h(n) \longrightarrow g(n) \text{ circularly convolved with } h(n)$$

* properties

$$g(n) \textcircled{N} h(n) = h(n) \textcircled{N} g(n)$$

$$(g(n) \textcircled{N} h(n)) \textcircled{N} w(n) = g(n) \textcircled{N} [h(n) \textcircled{N} w(n)]$$

circulant matrix

$$\begin{bmatrix} h(0) & h(N-1) & \dots & h(1) \\ h(1) & h(0) & \dots & h(2) \\ h(2) & h(1) & \dots & h(3) \\ \vdots & \vdots & \ddots & \vdots \\ h(N-1) & h(N-2) & \dots & h(0) \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \\ g(N-1) \end{bmatrix} = \begin{bmatrix} y_c(0) \\ y_c(1) \\ \vdots \\ y_c(N-1) \end{bmatrix}$$

* Circular convolution of 2 finite length sequences
(By using time reversal & Right shift)

Q.] Determine 4 point circular convolution for the
seq. $g[n] = \{1, 2, 0, 1\}$ $h[n] = \{2, 2, 1, 1\}$
 $0 \leq n \leq 3$

$$y_c(n) = g[n] \otimes h[n]$$

$$y_c[n] = g[n] \textcircled{4} h[n]$$

$$= \sum_{m=0}^3 g(m) h[\langle n-m \rangle_4] \quad \dots \quad \forall \cdot 0 \leq n \leq 3$$

$$y_c[0] = \sum_{m=0}^3 g(m) h[\langle 0-m \rangle_4] = \sum_{m=0}^3 g(m) h[\langle -m \rangle_4]$$

$$h[0], h[1], h[2], h[3]$$

$$h[3], h[2], h[1], h[0] \rightarrow \text{Time reversal.}$$

$$h[0], h[3], h[2], h[1] \rightarrow \text{Right shift}$$

$$h[n] = \{2, 2, 1, 1\}$$

$$\rightarrow h[\langle -m \rangle_4] = \{2, 1, 1, 2\}$$

$$y_c[0] = g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1]$$

$$= 1(2) + 2(1) + 0(1) + 1(2)$$

$$\underline{y_c[0] = 6}$$

$$h[\langle 1-m \rangle_4] = \{h[1], h[0], h[3], h[2]\}$$

$$\{2, 2, 1, 1\}$$

$$y_c[1] = g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2]$$

$$= 1(2) + 2(2) + 0(1) + 1(1)$$

$$\underline{y_c[1] = 7}$$

$$h[\langle 2-m \rangle_4] = \{1, 2, 2, 1\}$$

$$y_c[2] = 1(1) + 2(2) + 0(2) + 1(1)$$

$$\underline{y_c[2] = 6}$$

$$h[\langle 3-m \rangle_4] = \{1, 1, 2, 2\}$$

$$y_c[3] = 1(1) + 2(1) + 0(2) + 1(2)$$

$$\underline{y_c[3] = 5}$$

$$\therefore y_c[n] = \{6, 7, 6, 5\}$$

N-point circular convolution:

$$y_c(n) = \sum_{m=0}^{N-1} g[m] h[\langle n-m \rangle_N]$$

★ Calculate circular convolution using matrix method.

$$1) \quad x_1(n) = \{1, 3, 5, 3\} \quad x_2(n) = \{2, 3, 1, 1\}$$

$N=4$ $\uparrow$ $\uparrow$

$$x_3(m) = x_1(n) \cdot x_2[\langle m-m \rangle_N]$$

$$x_2(n) = \begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix}$$

$$x_2[\langle n-m \rangle_4] = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 3 & 2 \end{bmatrix}$$

$$x_3(n) = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 17 \\ 23 \\ 25 \end{bmatrix}$$

$x_3 = \{19, 17, 23, 25\}$

$$x_1(n) = \{ \underset{\uparrow}{2}, 1, 2, 1 \} \quad x_2(n) = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$$

$$x_3(n) = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

$$x_3 = \{ \underset{\uparrow}{14}, 16, 14, 16 \}$$

* Circular convolution property of DFT
 Suppose we have 2 finite duration sequences of length N , $x_1(n)$ & $x_2(n)$. DFT of 2 seq. are

$$X_1[k] = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N} \quad \forall k=0,1,2,\dots,N-1 \quad \text{--- (1)}$$

$$X_2[k] = \sum_{l=0}^{N-1} x_2(l) e^{-j2\pi lk/N} \quad \forall k=0,1,2,\dots,N-1 \quad \text{--- (2)}$$

$$\& \quad X_3[k] = X_1[k] \cdot X_2[k] \rightarrow \text{frequency domain seq.}$$

If we multiply 2 DFTs, the result is, say $X_3[k]$ & $X_3[k]$ is DFT of $x_3(n)$ of length N .

To determine the relationship b/w $x_1(n)$, $x_2(n)$ & $x_3(n)$

$$\text{We have } X_3[k] = X_1[k] \cdot X_2[k]$$

By IDFT

$$x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_3[k] e^{j2\pi nk/N}$$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} x_1[k] \cdot x_2[k] e^{j2\pi mk/N}$$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \left[\sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N} \right] \right.$$

$$\left. \left[\sum_{l=0}^{N-1} x_2(l) e^{-j2\pi lk/N} \right] \right\} e^{-j2\pi mk/N} \quad \text{--- (A)}$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} \right]$$

The inner sum in the bracket has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & \forall a=1 \\ \frac{1-a^N}{1-a} & \forall a \neq 1 \end{cases}$$

~~$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\frac{1 - e^{j2\pi k(m-n-l)}}{1 - e^{j2\pi k(m-n-l)/N}} \right]$$~~

We observe that if $(m-n-l)$ is multiple of N , then $a=1$

$$a = e^{j2\pi} = \cos 2\pi + j \sin 2\pi = 1$$

on the other hand $a^N = 1$ $\forall$ any value of $(m-n-l)$ which is multiple of N .

$$\therefore \sum_{k=0}^{N-1} a^k = \begin{cases} N & \forall ((m-n))_N \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore x_3(m) = \sum_{n=0}^{N-1} x_1(n) x_2((m-n))_N \quad \forall m=0,1,2 \dots N-1$$

Circular convolution in Time domain is equivalent to multiplication in frequency domain. $\xrightarrow{\text{Circular Convolution theorem}}$

$$g(n) \text{ (N) } h(n) \xleftrightarrow{\text{DFT}} G[k] H[k]$$

time domain frequency domain.

3 Q. Find circular convolution.

$$g[n] = \{1, 2, 0, 1\} \quad h[n] = \{2, 2, 1, 1\}$$

$\uparrow$ $\uparrow$

→ take DFT of $g[n] \rightarrow G[k]$
take DFT of $h[n] \rightarrow H[k]$

$$g[n] \text{ (N) } h[n] \leftrightarrow G[k] \cdot H[k]$$

$$G[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow G[k] = \{4, 1-j, -2, 1+j\}$$

$$G[k] = \{4, 1-j, -2, 1+j\}$$

$\uparrow$

$$H[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \{6, 1-j, 0, 1+j\}$$

$$H[k] = \{6, 1-j, 0, 1+j\}$$

$\uparrow$

$$G[k] \cdot H[k] = \{4(6), (1-j)(1-j), -2(0), (1+j)(1+j)\}$$

$$Y[k] = \{24, -2j, 0, 2j\}$$

IDFT of $Y[k]$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 24 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$

$$\frac{1}{4} \{24 - 2j + 0 + 2j\} = \underline{\underline{6}}$$

$$\frac{1}{4} \{24 + 2 + 0 + 2\} = \underline{\underline{7}} \quad \therefore y(n) = \{6, 7, 6, 5\}$$

$$\frac{1}{4} \{24 + 2j + 0 - 2j\} = \underline{\underline{6}}$$

$$\frac{1}{4} \{24 - 2 + 0 - 2\} = \underline{\underline{5}}$$

* For circulant matrix method.

$$\text{If } x_1(n) = \{1, -1, -2, 3, -1\} \quad N=5$$

$$\& x_2(n) = \{1, 2, 3\} \quad N=3$$

$$\rightarrow \therefore x_2(n) = \{1, 2, 3, 0, 0\}$$

$$x_3(n) = \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

$$x_3(n) = \{8, -2, -1, -4, -1\}$$

By using matrix method. Use DFT property of circular convolution method & calculate.

$$x(n) = \{1, 2\}_{(N=2)} \quad \text{Evaluation of linear filtering using DFT}$$

$$h(n) = \{2, 2\}_{(M=2)} \rightarrow \text{Ans samples} \Rightarrow (N+M-1)$$

$$X[k] = \begin{bmatrix} 1 & 1 \\ 1 & -j \\ 1 & -1 \\ 1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \{3, 1-2j, -1, 1+2j\}$$

$$h[k] = \begin{bmatrix} 1 & 1 \\ 1 & -j \\ 1 & -1 \\ 1 & j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \{4, 2-2j, 0, 2+2j\}$$

$$X[k] \cdot h[k] = \{12, (1-2j)(2-2j), 0, (1+2j)(2+2j)\}$$

$$= \{12, -2-6j, 0, -2+6j\}$$

$$y[n] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 12 \\ -2-6j \\ 0 \\ -2+6j \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ 24 \\ 16 \\ 0 \end{bmatrix}$$

$$y[n] = \{2, 6, 4\}$$

Implementation of FFT algorithm for IIR filter (system).

Linear convolution finite length seq.

(length of one sequence is infinite or very much greater than the second sequence)

eg → processing of speech signal.

→ overlap add method

→ overlap & save method

} these 2 methods are used for processing these type of sequences.

Let $h(n)$ is finite length seq. of length M & $x(n)$ is infinite length sequence.

→ Objective is to develop computationally efficient method (which is DFT based method) to implement the linear convolution b/w $x[n]$ & $h[n]$.

$$y[n] = \sum_{l=0}^{M-1} h[l]x[n-l] = h[n] * x[n]$$

★ Overlap add method

Size of each data block is N

Each I/P data block is formed by taking L samples from I/P sequence & adding $M-1$ zeros at the end of each I/P sequence.

$$x_1(n) = \underbrace{\{x(0), x(1), \dots, x(L-1)\}}_{\text{First } L \text{ samples from I/P Sequence}} \underbrace{\{0, 0, 0, \dots\}}_{(M-1) \text{ zeros}}$$

$$x_2(n) = \underbrace{\{x(L), x(L+1), \dots, x(2L-1)\}}_{\text{Next } L \text{ samples}} \underbrace{\{0, 0, 0, \dots\}}_{(M-1) \text{ zeros}}$$

$$x_3(n) = \underbrace{\{x(2L), x(2L+1), \dots, x(3L-1)\}}_{\text{Next } L} \underbrace{\{0, 0, 0, \dots\}}_{(M-1) \text{ zeros}}$$

Impulse response of filter is $h(n)$. It's length is M

$$h(n) = \underbrace{\{h(0), h(1), \dots, h(M-1)\}}_{M \text{ samples of I/p sequence}} \underbrace{\{0, 0, 0, \dots, 0\}}_{(L-1) \text{ zeros}}$$

eg $x(n) = \{1, 3, 9, 10, 0, 5, 6, 7\}$ $L=3, M=2$

$$x_1(n) = \{1, 3, 9, 0\} \quad x_2(n) = \{10, 0, 5, 0\} \quad x_3(n) = \{6, 7, 0\}$$

- Now DFT of each data block is calculated separately & represented as $X(k)$

- DFT of $h(n)$, $H(k)$ is calculated.

- O/p of each block is calculated

$$Y_m(k) = H(k) \cdot X_m(k) \quad m = 1, 2, 3, \dots$$

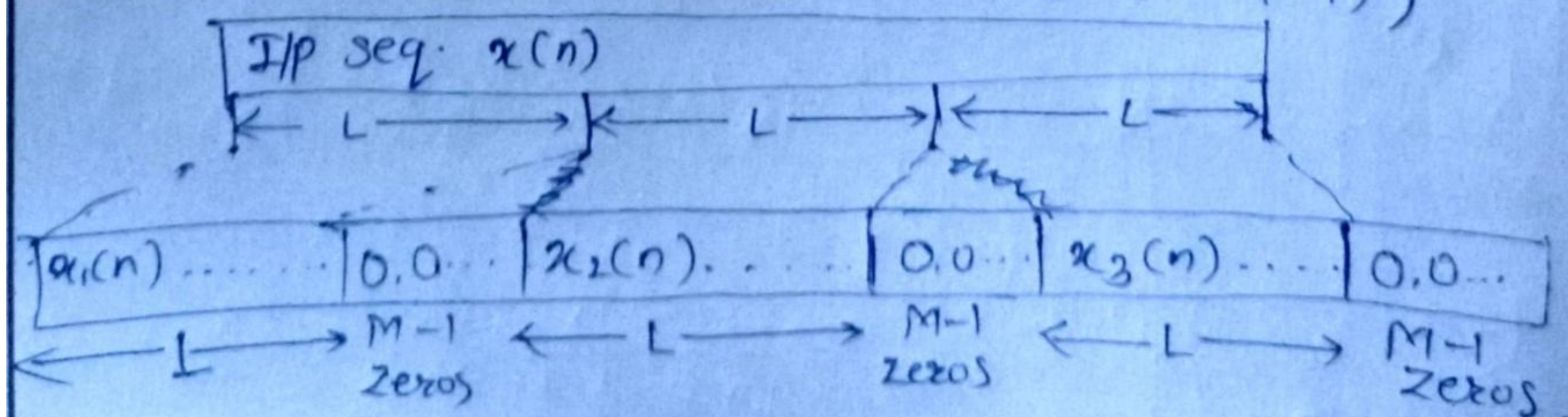
$$Y_1(k) = H(k) \cdot X_1(k)$$

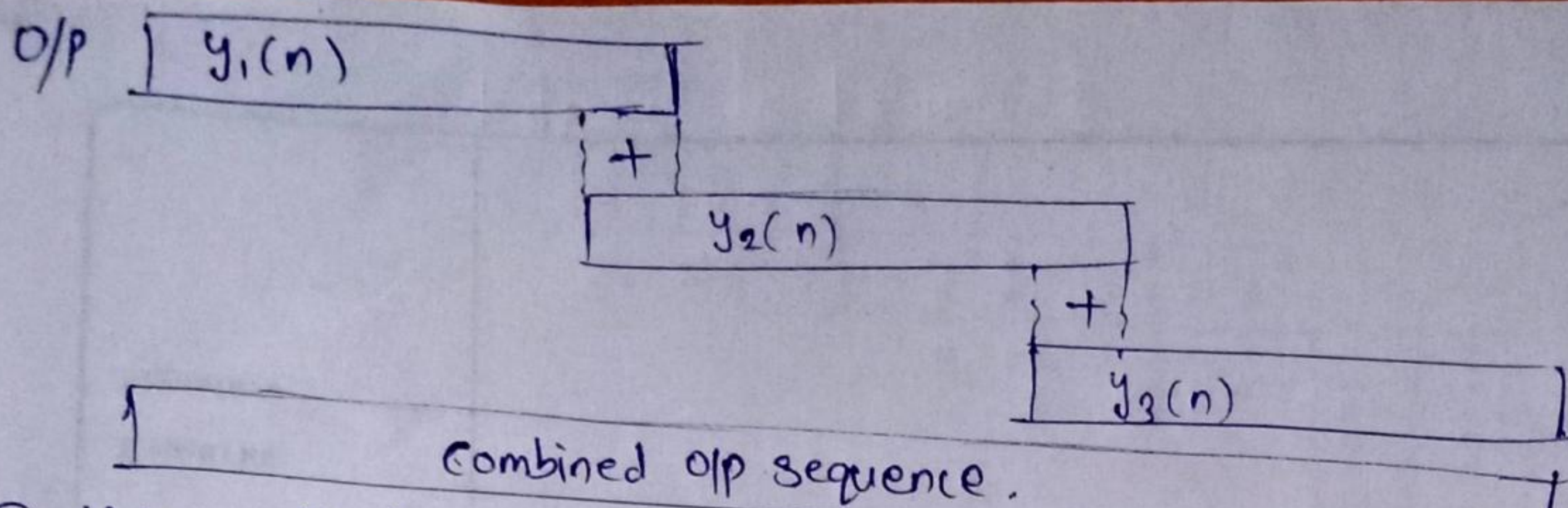
- $y(n)$ is obtained by taking IDFT of $Y_m(k)$

- Here, we are performing N -point DFT for each block. Different o/p sequences are as follows.

$$y_1(n) = \{y_1(0), y_1(1), \dots, y_1(L-1), y_1(L), y_1(L+1), \dots, y_1(L+M-1)\}$$

$$y_2(n) = \{y_2(0), y_2(1), \dots, y_2(L-1), y_2(L), y_2(L+1), \dots, y_2(L+M-1)\}$$





In this method each I/p sequence is padded with $(M-1)$ zeros at the end to make the length $N = L + M - 1$

So last $(M-1)$ samples of each o/p block must be overlapped & added to first $M-1$ samples of next o/p block. Hence the name overlap & add method.

Q.] $x(n) = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$
 $h(n) = \{1, 2, 3\} \rightarrow (M=3)$

→ Let length of block = 4 ($N=4$)
 $M-1 = 2$

Each block is obtained by adding 'M-1' no. of zeros at the end of each block.

$$h(n) = \{1, 2, 3, 0\}$$

$$x_1(n) = \{1, 2, 0, 0\} \quad N=4, M-1=2$$

$$x_2(n) = \{-1, 2, 0, 0\}$$

$$x_3(n) = \{3, -2, 0, 0\}$$

$$x_4(n) = \{-3, -1, 0, 0\}$$

$$x_5(n) = \{1, 1, 0, 0\}$$

$$x_6(n) = \{2, -1, 0, 0\}$$

$$y_1(n) = x_1(n) \otimes h(n) \quad \text{time domain}$$

$$y_1(n) = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 6 \end{bmatrix} = \{1, 4, 7, 6\}$$

$$y_2(n) = \{-1, 0, 1, 6\} \quad y_5(n) = \{1, 3, 5, 3\}$$

$$y_3(n) = \{3, 4, 5, -6\} \quad y_6(n) = \{2, 3, 4, -3\}$$

$$y_4(n) = \{3, 5, 7, -3\} \quad y_4(n) = \{-3, -7, -11, -3\}$$

$$\begin{aligned}
 y_1(n) &= [1, 4, 7, 6] \\
 y_2(n) &= [-1, 0, 1, 6] \\
 y_3(n) &= [3, 4, 5, -6] \\
 y_4(n) &= [-3, -7, -11, -3] \\
 y_5(n) &= [1, 3, 5, 3] \\
 y_6(n) &= [2, 3, 4, -3]
 \end{aligned}$$

$$y(n) = \{1, 4, 6, 6, 4, 10, 2, -13, -10, 0, 7, 6, 4, -3\}$$

★ Overlap & Save method

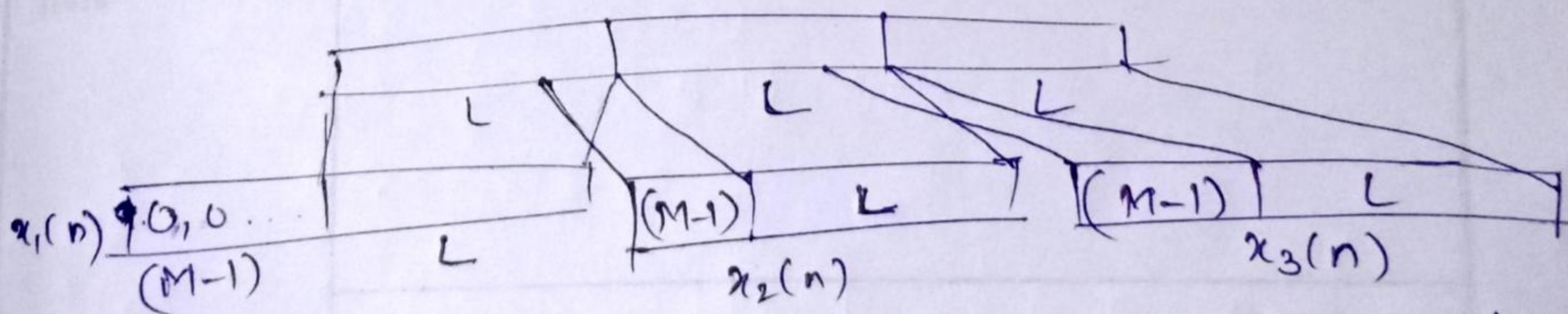
Each block of length N is formed by taking ' L ' samples of current segment & ' $M-1$ ' no. of samples from previous segment. ($N = L + M - 1$)

For 1st block $x_1(n)$, there is no previous segment. $\therefore$ This block is padded with $(M-1)$ zeros

$$\therefore x_1(n) = \{ \underbrace{0, 0, \dots, 0}_{(M-1)}, \underbrace{x(0), x(1), \dots, x(L-1)}_{\text{first } L \text{ samples of } x(n)} \}$$

$$x_2(n) = \{ \underbrace{x(L-M+1), \dots, x(L-1)}_{(M-1) \text{ samples of previous segment}}, \underbrace{x(L), x(L+1), \dots, x(2L-1)}_{x(2L-1)} \}$$

$$x_3(n) = \{ \underbrace{x(2L-M+1), \dots, x(2L-1)}_{(M-1) \text{ samples of previous seg.}}, \underbrace{x(2L), x(2L+1), \dots, x(3L-1)}_{x(3L-1)} \}$$



length of impulse response is M , N should be equal to M

$\therefore$ Append $(L-1)$ zeros in $h(n)$

$$h(n) = \{ h(0), h(1), \dots, h(M-1), \underbrace{0, 0, \dots, 0}_{(L-1) \text{ zeros}} \}$$

$$y_m(k) = H(k) \cdot X_m(k)$$

taking IDFT of $Y_m(k)$

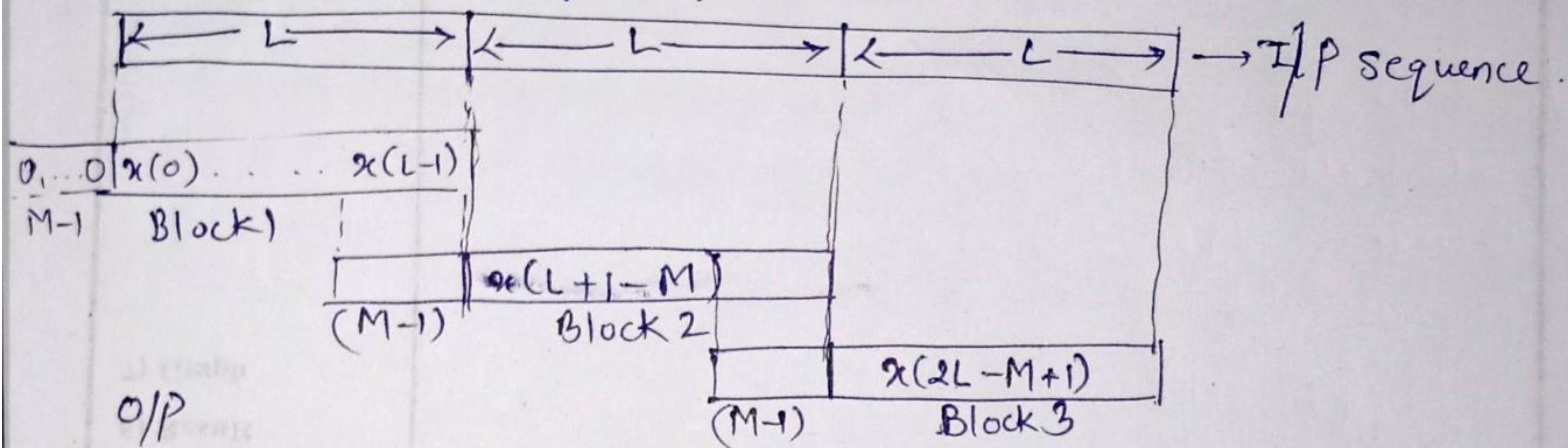
$$Y_m(n) = \text{IDFT} \{ H(k) \cdot X_m(k) \}$$

Find DFT of each block $X_m(k)$ DFT of $x_m(n)$ $m=1, 2, 3 \dots$

$$\hat{Y}_m(k) = H(k) X_m(k)$$

$$y(n) = \text{IDFT of } \hat{Y}_m(k)$$

In this case, in o/p block due to overlapping aliasing occurs. So discard first $(M-1)$ samples of $y_m(n)$



Q. $x(n) = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$
 $h(n) = \{1, 2, 3\}$ Make use of overlap & save method to find the response.
 $N=4, M=3 \Rightarrow M-1=2$
 $h(n) = \{1, 2, 3, 0\} \therefore M=4$
 $x_1(n) = \{0, 0, 1, 2\}$ $x_2(n) = \{1, 2, -1, 2\}$
 $x_3(n) = \{-1, 2, 3, -2\}$ $x_4(n) = \{3, -2, -3, -1\}$
 $x_5(n) = \{-3, -1, 1, 1\}$ $x_6(n) = \{1, 1, 2, -1\}$ $x_7(n) = \{2, -1, 0, 0\}$

$$\therefore y_1(n) = x_1(n) \textcircled{4} h(n) = \{7, 6, 1, 4\}$$

$$y_2(n) = \{2, 10, 6, 6\}$$

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$$y_3(n) = \{4, -6, 4, 10\} \quad y_4(n) = \{+8, 1, 2, -13\}$$

$$y_5(n) = \{2, -4, -10, 0\} \quad y_6(n) = \{5, 0, 7, 6\} \quad y_7(n) = \{2, 2, 4, -3\}$$

$$y(n) = \{1, 4, 6, 6, 4, 10, 2, -13, -10, 0, 7, 6, 4, -3\}$$

* Structures of Discrete time systems.

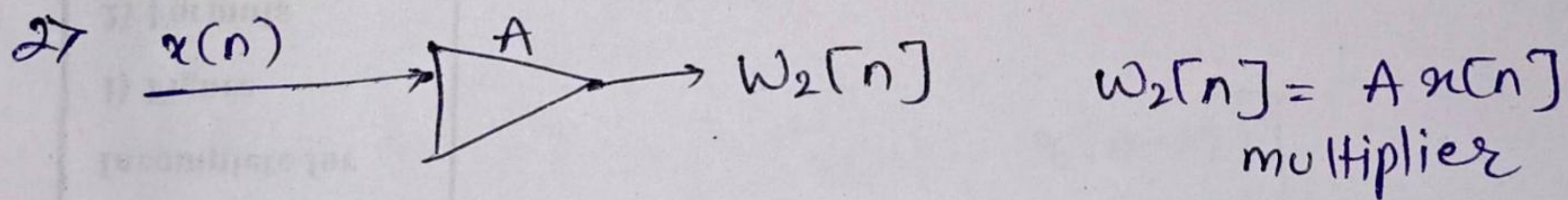
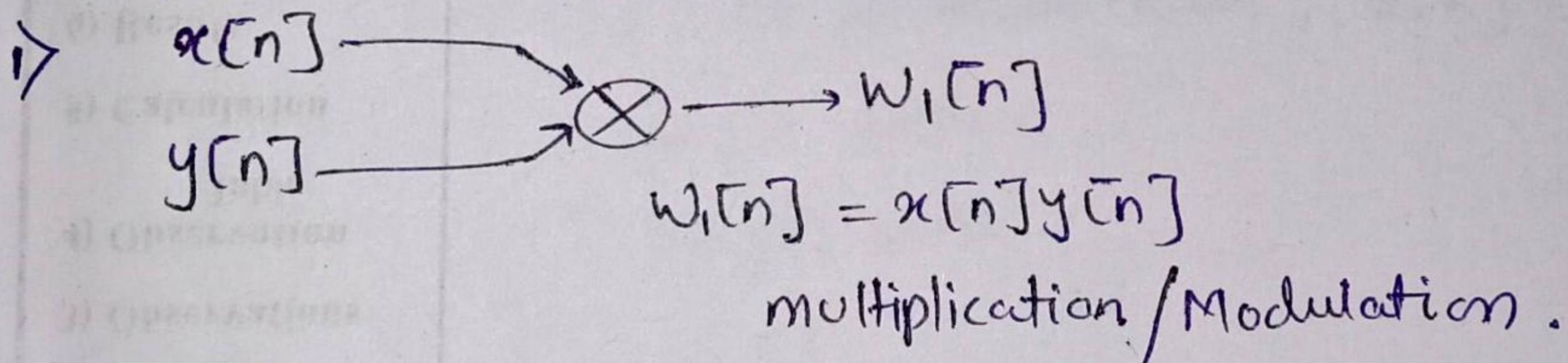
function → to process ^{given} discrete time signal & generate another discrete time sequence (o/p sequence) → I/p sequence.

In practical digital systems all signals are digital signals & operation of signal will also lead to digital signal.

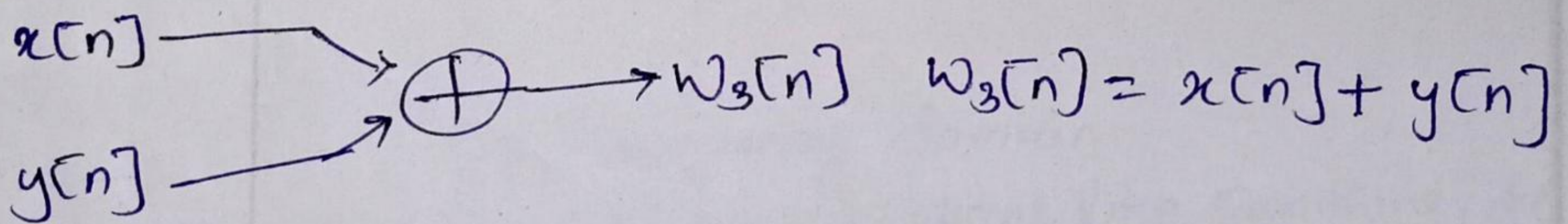
These discrete time systems are called as "Digital Filter".

$$x(n) \rightarrow [H(\cdot)] \rightarrow y(n) = H(x(n))$$

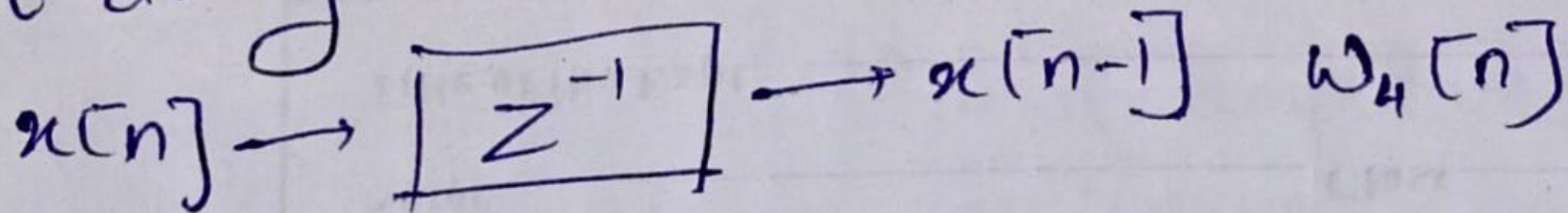
Representation of discrete time system.



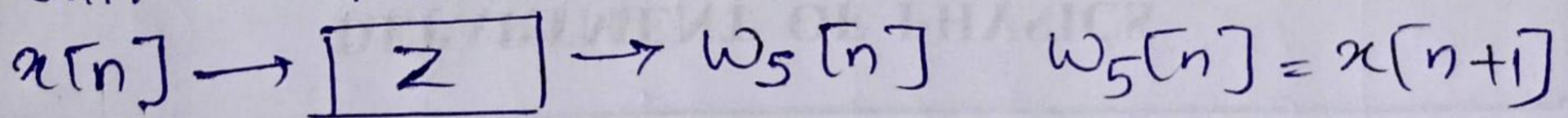
3) Adder



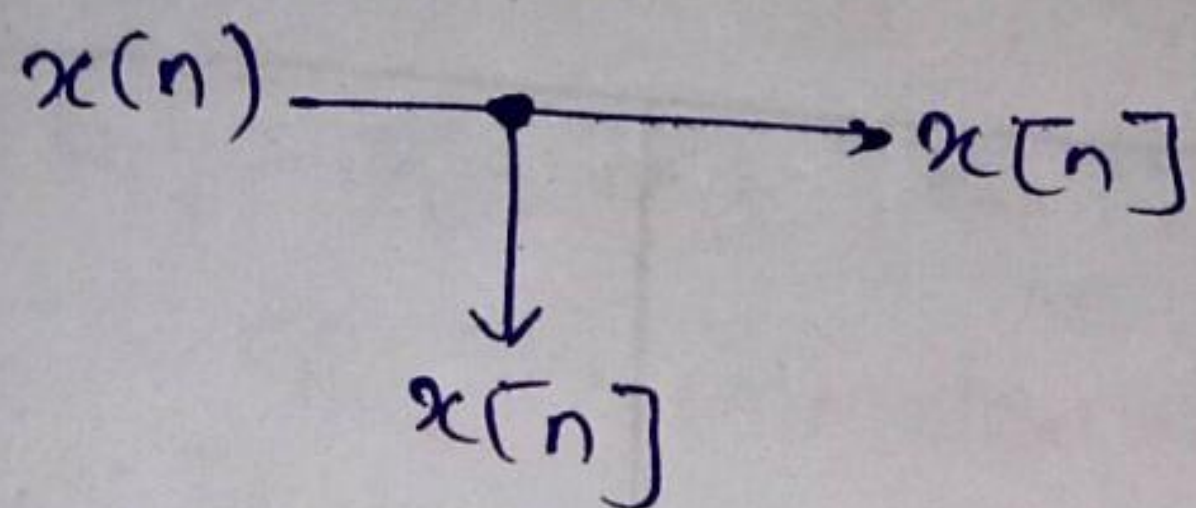
4) Unit delay



5) Unit advance



6] Pick off node



Elementary Discrete time Systems

1) Accumulator

$$y[n] = \frac{1}{M} \sum_{l=0}^{M-1} x[n-l]$$

The o/p $y[n]$ at time instant 'n' is the sum of the I/p samples $x[n]$ & all past I/p samples from $-a$ to n .

2) moving average filter

cannot measure data accurately due to random variations in measurements, due to additive noise data gets corrupted.

The n^{th} sample of data is modeled as, $x[n] = s[n] + d[n]$

$s[n]$ = n^{th} sample data.

$d[n]$ = n^{th} sample noise.

$$y[n] = \frac{1}{M} \sum_{l=0}^{M-1} x[n-l]$$

$$y[n] = y[n-1] + \frac{1}{M} [x[n] - x[n-M]]$$

M point moving average filter.

Filter classification

1) Time domain 2) Frequency domain

Time domain filters are used for operations like smoothing, dc removal & waveform shaping.

When information is in the form of amplitude, frequency & phase, we need to separate the frequency bands. Here we use frequency domain filter.

Eg → Smoothing.

Consider 5 point moving average filter.

$$y[80] = x[80] + x[81] + x[82] + x[83] + x[84] / 5$$

This is one side averaging.

OR $y[80] = x[78] + x[79] + x[80] + x[81] + x[82] / 5$

This symmetrical averaging. Probability of use of this method is more.

* Digital Filters can be implemented in 2 ways.
1) By convolution: called as finite impulse response (FIR)
2) By recursion: called as infinite impulse response (IIR)

* convolution: better performance, execute slowly. convolving I/P $x(n)$ with digital filter's impulse response $h(n)$.

FIR: when impulse response is used in this way, it is called as filter kernel.

gaining o/p by weighing the samples in the I/P & adding them together.

IIR: Instead of using filters, recursive filters are used defined by a set of recursion coefficients.

* LTI system & difference equation:

The linear constant coefficient difference equation is expressed as,

$$y(n) + a_1 y(n-1) + \dots + a_N y(n-N) =$$

$$b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M)$$

$$y(n) = [b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M)] -$$

$$[a_1 y(n-1) + \dots + a_N y(n-N)]$$

$$y(n) = \sum_{k=0}^M b_k x(n-k) - \sum_{k=1}^N a_k y(n-k)$$

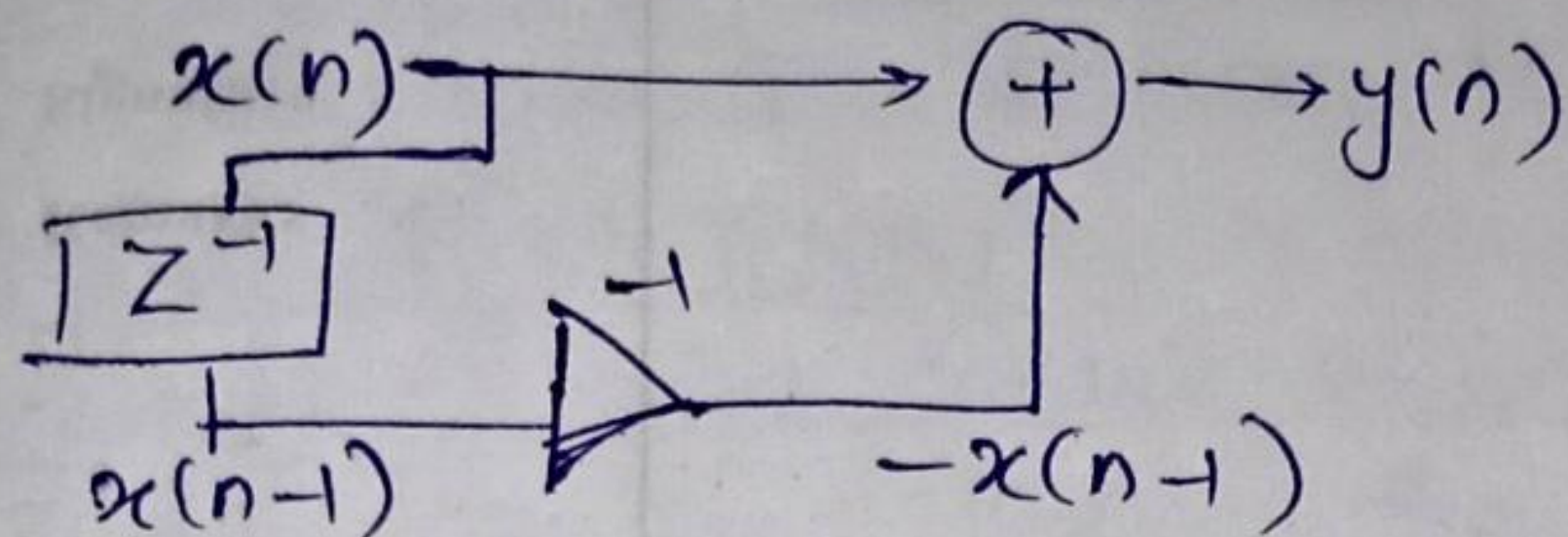
where b_k = delay dependent feed forward coefficient.

a_k = delay dependent feedback coefficient.

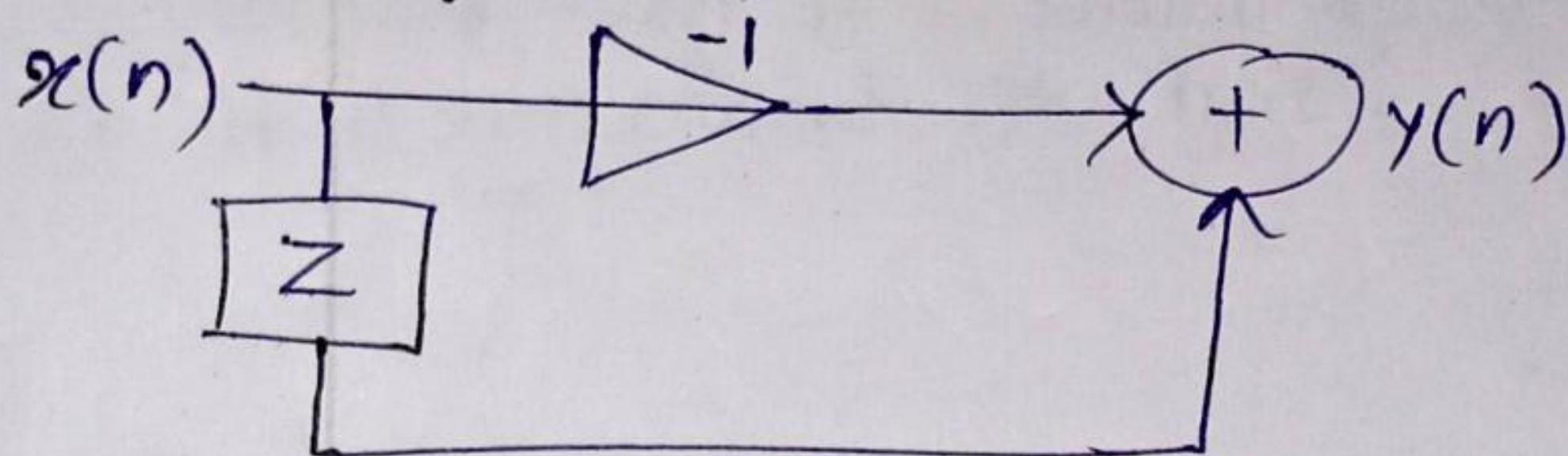
Block diagram representation

LTI systems $\rightarrow$ I/p-o/p given by difference equation with inter connection $\rightarrow$ of basic building blocks.

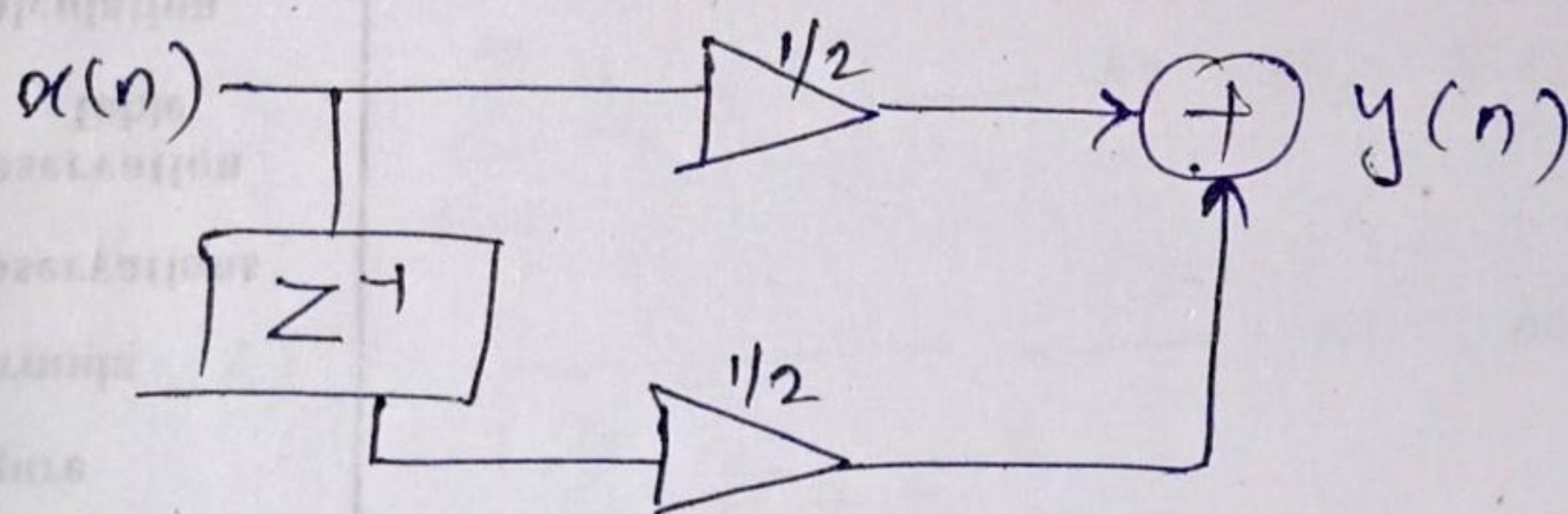
① $y(n) = x(n) - x(n-1]$



② $y(n] = x(n+1] - x(n]$



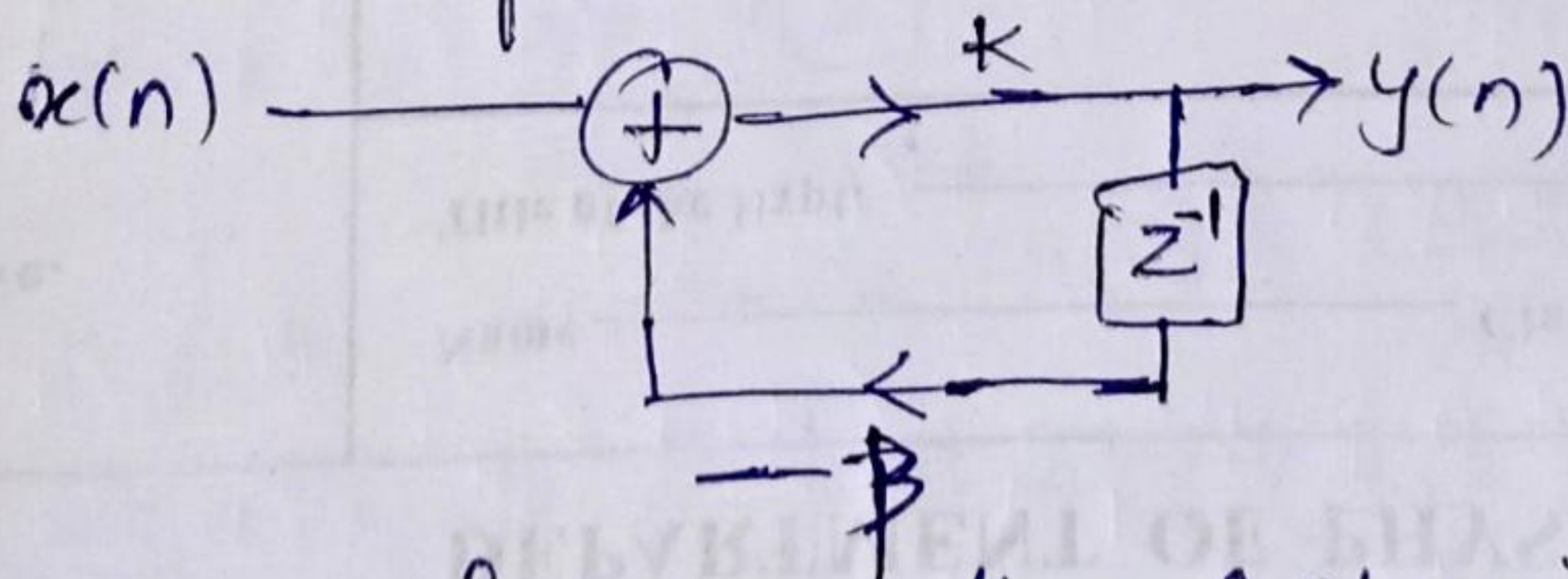
③ $y(n] = \frac{x(n] + x(n-1]}{2} = \frac{1}{2}x(n] + \frac{1}{2}x(n-1]$



For recursive LTI systems, $y(n] = K[x(n] - \beta y(n-1)]$

- o/p sample is ~~used~~ scaled by β & fed back to I/p.
- I/p affects the system thru 2 paths.:

1) Direct. 2) feedback.



* Advantages of representing system in block diagram.

1) Computation algorithm can be easily written.

2) with the help of transfer fn. variety of equivalent block

- diagram representation can be developed.
- 3) Hardware requirements can be easily detected.
 - 4) Relationship b/w I/P & O/P can be easily established.

* various types of structure.

1) Direct form I 2) Direct form II 3) Cascade 4) Parallel

types of structure

canonic & non canonic structures.

If the no. of delays in the basic realization block diagram is equal to the order of difference equation or order of transfer functions then it is known as canonic structure otherwise it is non canonic structure.

Arrang eqⁿ ② $\xrightarrow{\text{transfer fn. (numerator of sys.)}}$

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = \sum_{k=0}^M b_k z^{-k} \text{ — All zero system.}$$

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^N a_k z^{-k}} \text{ — All pole system.}$$

Direct form 1.

$$H_1(z) = \sum_{k=0}^M b_k z^{-k}$$

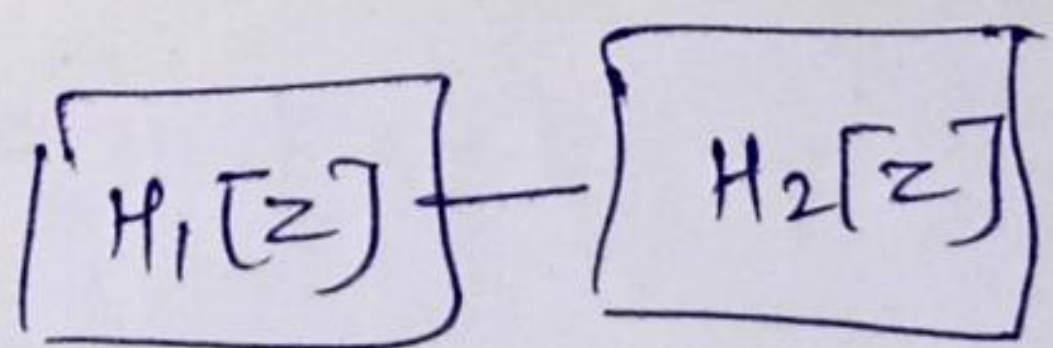
$$H_1(z) = \frac{\text{o/p } Y[z]}{\text{i/p } X[z]}$$

Let us write $H_1(z) = \frac{W[z]}{X[z]}$

$W[z] = \text{o/p of first stage } (H_1(z))$

$$\frac{W[z]}{X[z]} = \sum_{k=0}^M b_k z^{-k}$$

$$\frac{W[z]}{X[z]} = b_0 z^0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}$$



$$z^0 = 1$$

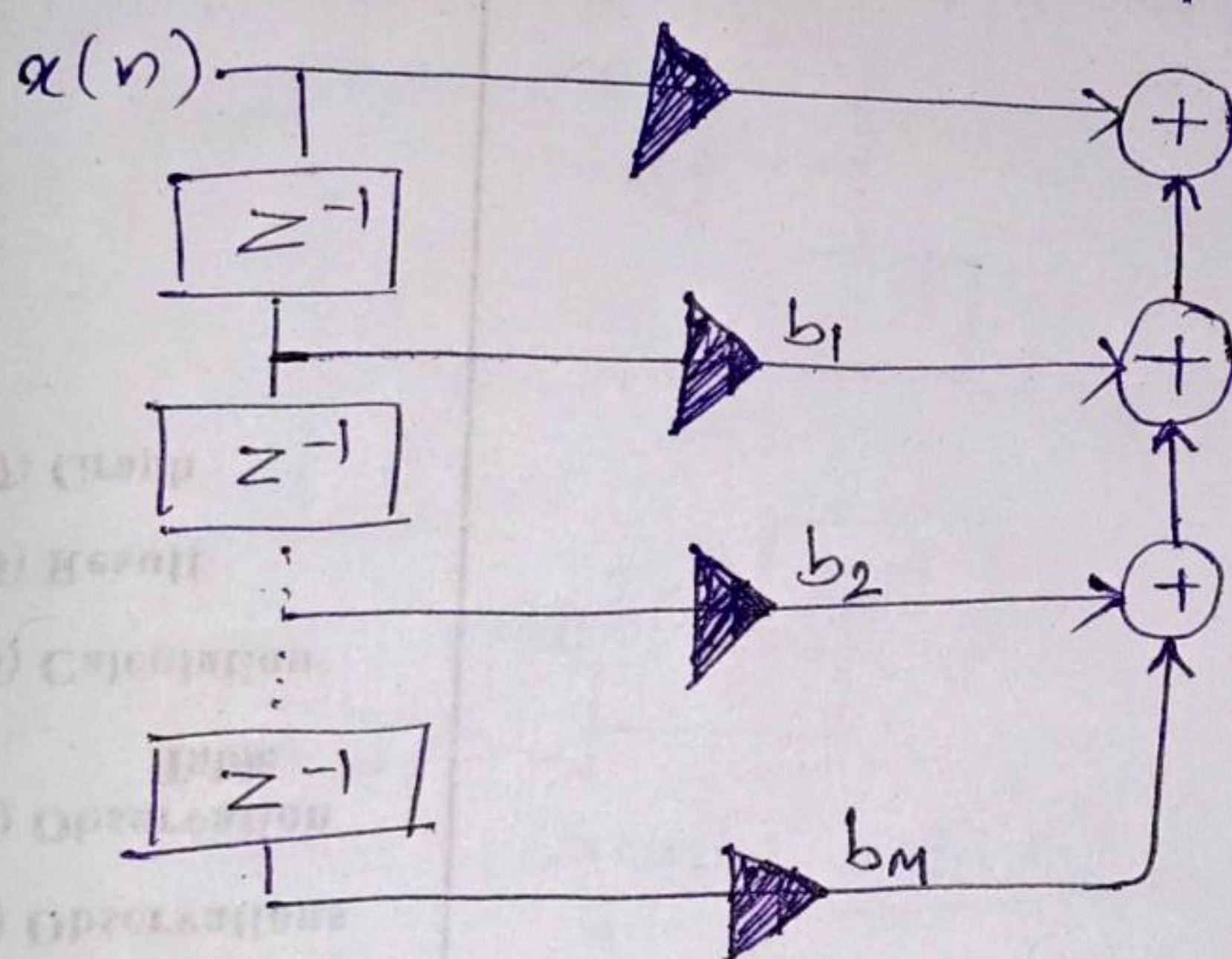
$$\frac{W(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m}$$

$$W(z) = X(z) [b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m}]$$

$$W(z) = X(z) b_0 + X(z) b_1 z^{-1} + X(z) b_2 z^{-2} + \dots + X(z) b_m z^{-m}$$

Take inverse z transform

$$w(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) + \dots + b_m x(n-m)$$



This is.
Direct form
realization of
 $H_1(z)$

$$H_2(z) = \frac{\text{O/P } Y(z)}{\text{I/P } W(z)} = \frac{1}{1 + \sum_{k=1}^N a_k z^{-k}} = \frac{Y(z)}{W(z)}$$

$$Y(z) \left[1 + \sum_{k=1}^N a_k z^{-k} \right] = W(z)$$

$$Y(z) + Y(z) \sum_{k=1}^N a_k z^{-k} = W(z)$$

$$Y(z) = W(z) - Y(z) \sum_{k=1}^N a_k z^{-k}$$

$$Y(z) = - \left[\sum_{k=1}^N a_k z^{-k} \right] Y(z) + W(z)$$

$$Y(z) = - [a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}] Y(z) + W(z)$$

$$\therefore Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \dots - a_N z^{-N} Y(z) + W(z)$$

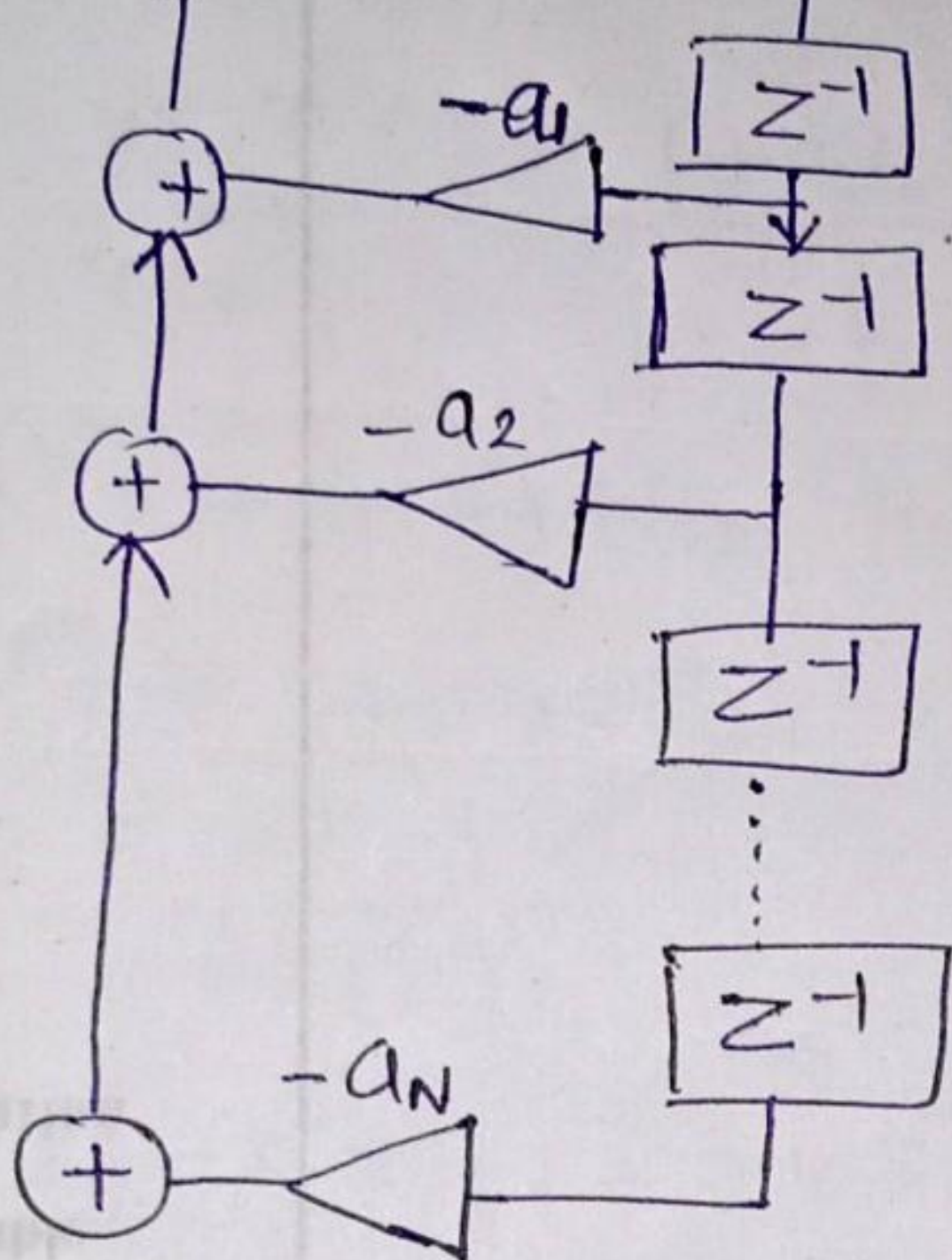
Taking inverse z transform

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \dots - a_N y(n-N) + w(n)$$

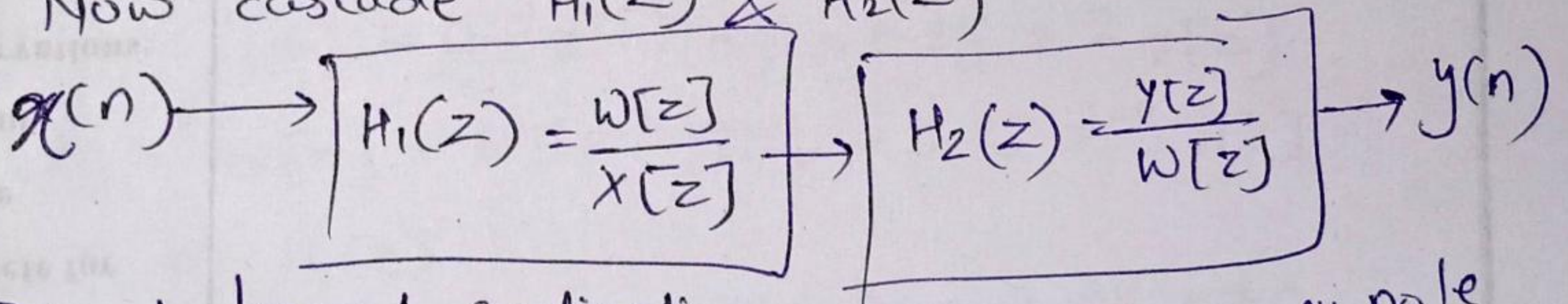
$-a_1, -a_2, \dots, -a_N$ are the coefficients.

Here all terms except $w(n)$ are delayed outputs hence it is a feedback connections.

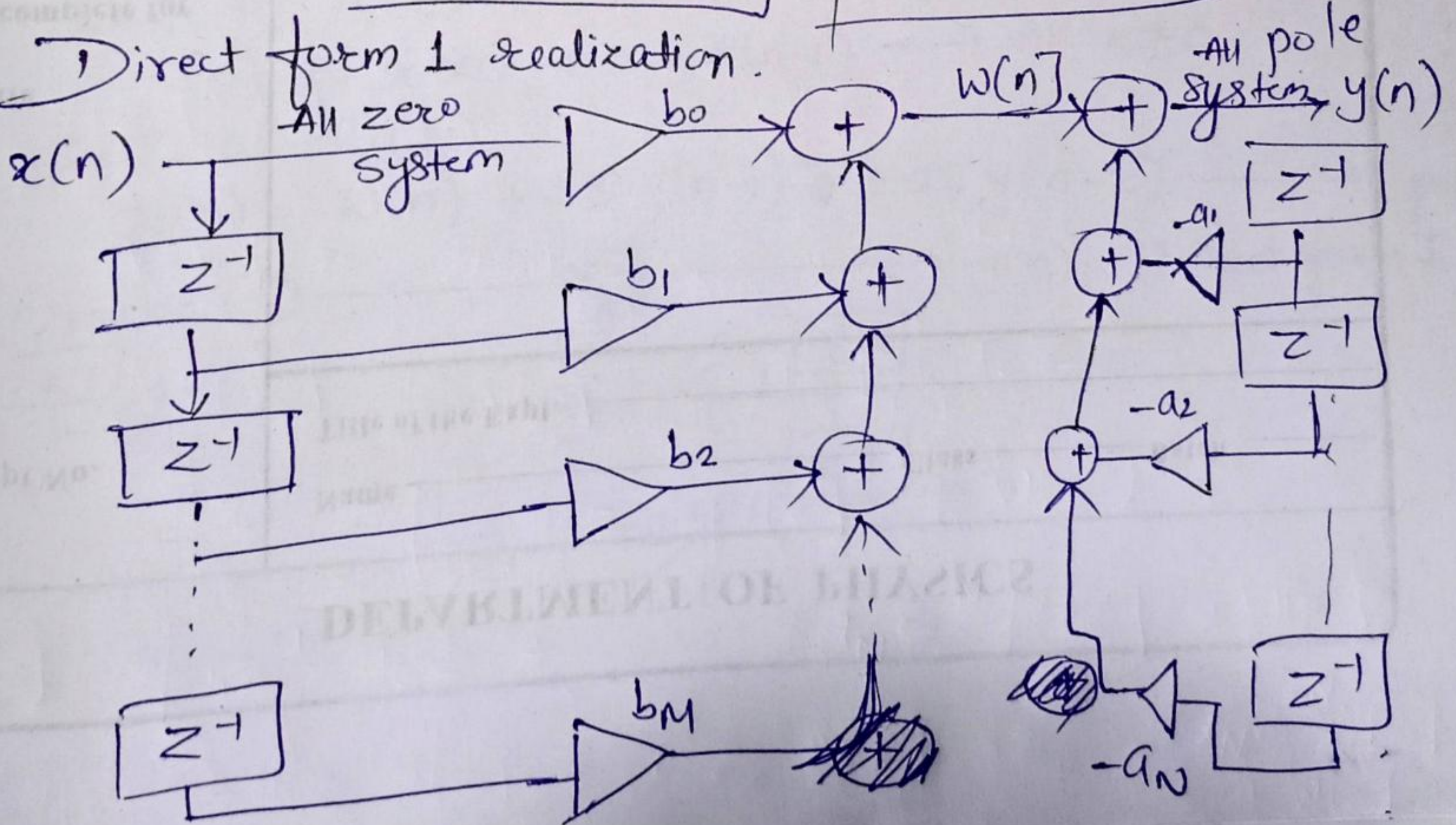
$w(n) \rightarrow y(n)$ All pole system



Now cascade $H_1(z)$ & $H_2(z)$



Direct form 1 realization.



$$8] y(n) = 0.5 y(n-1) - 0.25 y(n-2) + \underbrace{x(n) + 0.4 x(n-1)}_{w(n)}$$

→ take z transform of this eqⁿ.

$$\therefore Y[z] = 0.5 Y[z] z^{-1} - 0.25 Y[z] z^{-2} + X[z] + 0.4 X[z] z^{-1}$$

$$Y[z] - Y[z] 0.5 z^{-1} + 0.25 Y[z] z^{-2} = X[z] + 0.4 X[z] z^{-1}$$

$$Y[z] [1 - 0.5 z^{-1} + 0.25 z^{-2}] = X[z] [1 + 0.4 z^{-1}]$$

$$H[z] = \frac{Y[z]}{X[z]} = \frac{1 + 0.4 z^{-1}}{1 - 0.5 z^{-1} + 0.25 z^{-2}}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = 1 + 0.4 z^{-1}$$

$$= \frac{W[z]}{X[z]}$$

$$H_2(z) = \frac{1}{1 - 0.5 z^{-1} + 0.25 z^{-2}}$$

$$= \frac{Y[z]}{W[z]}$$

$$\therefore W[z] = X[z] [1 + 0.4 z^{-1}]$$

$$W[z] = Y[z] [1 - 0.5 z^{-1} + 0.25 z^{-2}]$$

$$W[z] = Y[z] - 0.5 z^{-1} Y[z] + 0.25 z^{-2} Y[z]$$

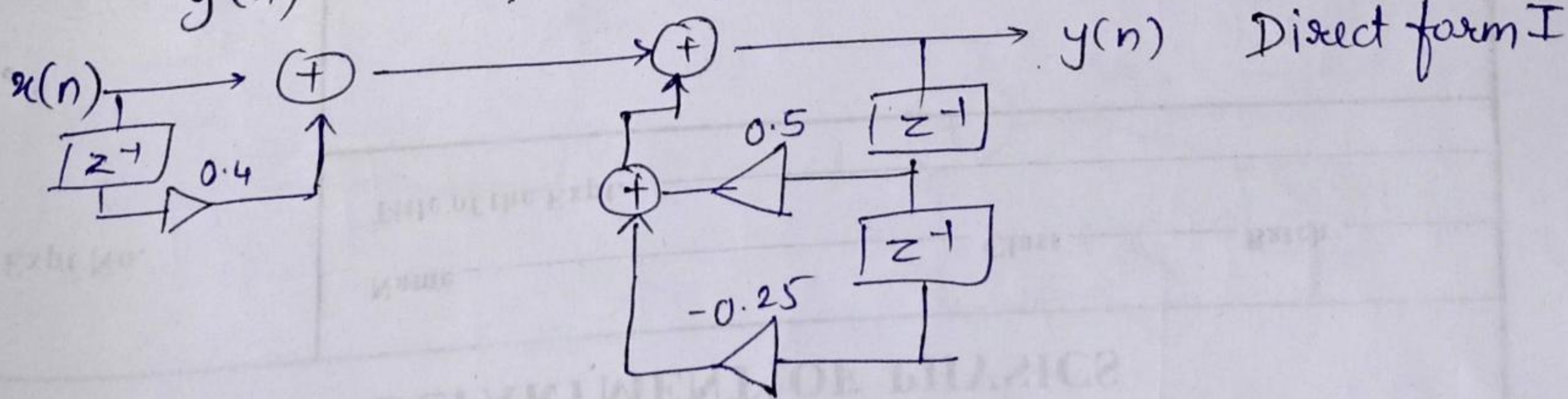
$$Y[z] = W[z] + 0.5 z^{-1} Y[z] - 0.25 z^{-2} Y[z]$$

IZT of $H_1(z)$

$$w(n) = x(n) + 0.4 x(n-1) \rightarrow \text{All zero}$$

IZT of $H_2(z)$

$$y(n) = w(n) + 0.5 y(n-1) - 0.25 y(n-2) \rightarrow \text{All pole}$$



Q. Obtain Direct form realization for
 $y(n) - 3/4 y(n-1) + 1/8 y(n-2) = x(n) + 1/2 x(n-1)$

→ Take Z.T.

$$Y[z] - 3/4 Y[z]z^{-1} + 1/8 Y[z]z^{-2} = X[z] + 1/2 X[z]z^{-1}$$

$$Y[z] \left[1 - 3/4 z^{-1} + 1/8 z^{-2} \right] = X[z] \left[1 + 1/2 z^{-1} \right]$$

$$H[z] = \frac{Y[z]}{X[z]} = \frac{1 + 1/2 z^{-1}}{1 - 3/4 z^{-1} + 1/8 z^{-2}}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = 1 + 1/2 z^{-1}$$

$$= \frac{W[z]}{X[z]}$$

$$H_2(z) = \frac{1}{1 - 3/4 z^{-1} + 1/8 z^{-2}}$$

$$= \frac{Y[z]}{W[z]}$$

$$W[z] = X[z] \left[1 + 1/2 z^{-1} \right]$$

$$W[z] = Y[z] \left[1 - 3/4 z^{-1} + 1/8 z^{-2} \right]$$

$$W[z] = Y[z] - Y[z] 3/4 z^{-1} + Y[z] 1/8 z^{-2}$$

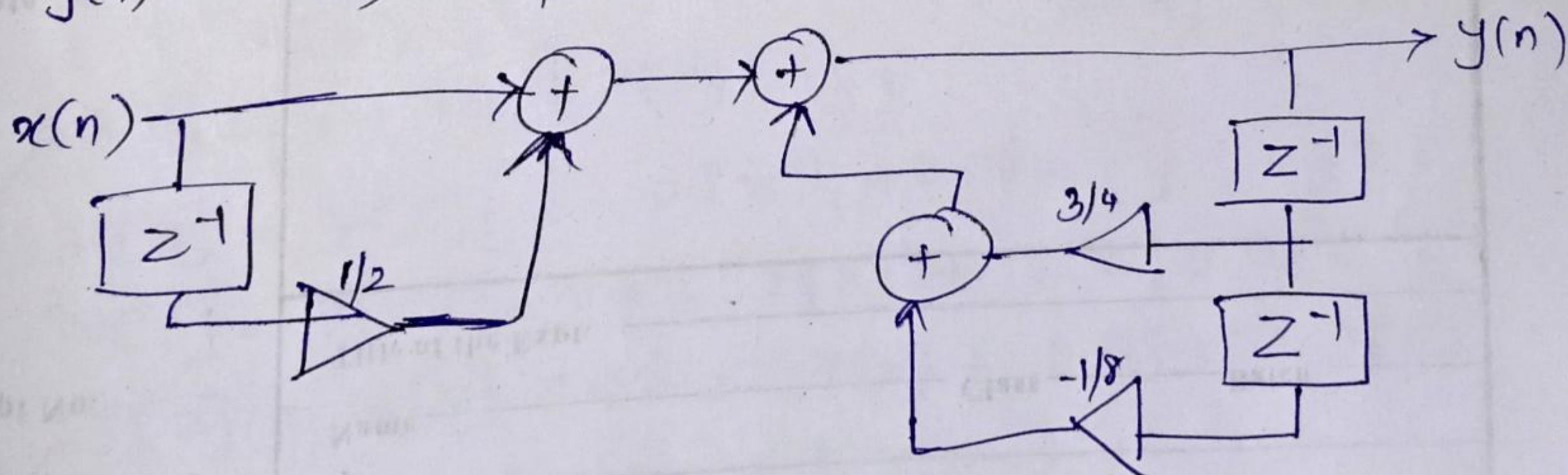
$$Y[z] = W[z] + Y[z] 3/4 z^{-1} - Y[z] 1/8 z^{-2}$$

IZT of $H_1(z)$

$$w(n) = x(n) + \frac{1}{2} x(n-1)$$

IZT of $H_2(z)$

$$y(n) = w(n) + 3/4 y(n-1) - 1/8 y(n-2)$$



Q.1] A certain discrete time LTI filter is having following data

1) Poles at 0.2 & 0.4

2) zeros at -0.4 & origin

3) Gain of filter is 5

Determine Direct form I realization.

→ zeros at $z_1 = 0$ & $z_2 = -0.4$

Poles at $p_1 = 0.2$ & $p_2 = 0.6$

Gain = 5

$$H(z) = \frac{k(z-z_1)(z-z_2) \dots (z-z_n)}{(z-p_1)(z-p_2) \dots (z-p_n)}$$

$$H(z) = \frac{5(z-0)(z+0.4)}{(z-0.2)(z-0.6)}$$

$$= \frac{5(z^2+0.4z)}{z^2-0.6z-0.2z+0.12} = \frac{5(z^2+0.4z)}{z^2-0.8z+0.12}$$

$$= \frac{5z^2+2z}{z^2-0.8z+0.08} = \frac{z^2(5+2z^{-1})}{z^2(1-\frac{0.8}{z}+\frac{0.08}{z^2})}$$

$$\frac{Y(z)}{X(z)} = \frac{5+2z^{-1}}{1-0.8z^{-1}+0.08z^{-2}}$$

$$H_1(z) = 5+2z^{-1} = \frac{W(z)}{X(z)}$$

$$H_2(z) = \frac{1}{1-0.8z^{-1}+0.08z^{-2}} = \frac{Y(z)}{W(z)}$$

$$W(z) = [5+2z^{-1}] X(z)$$

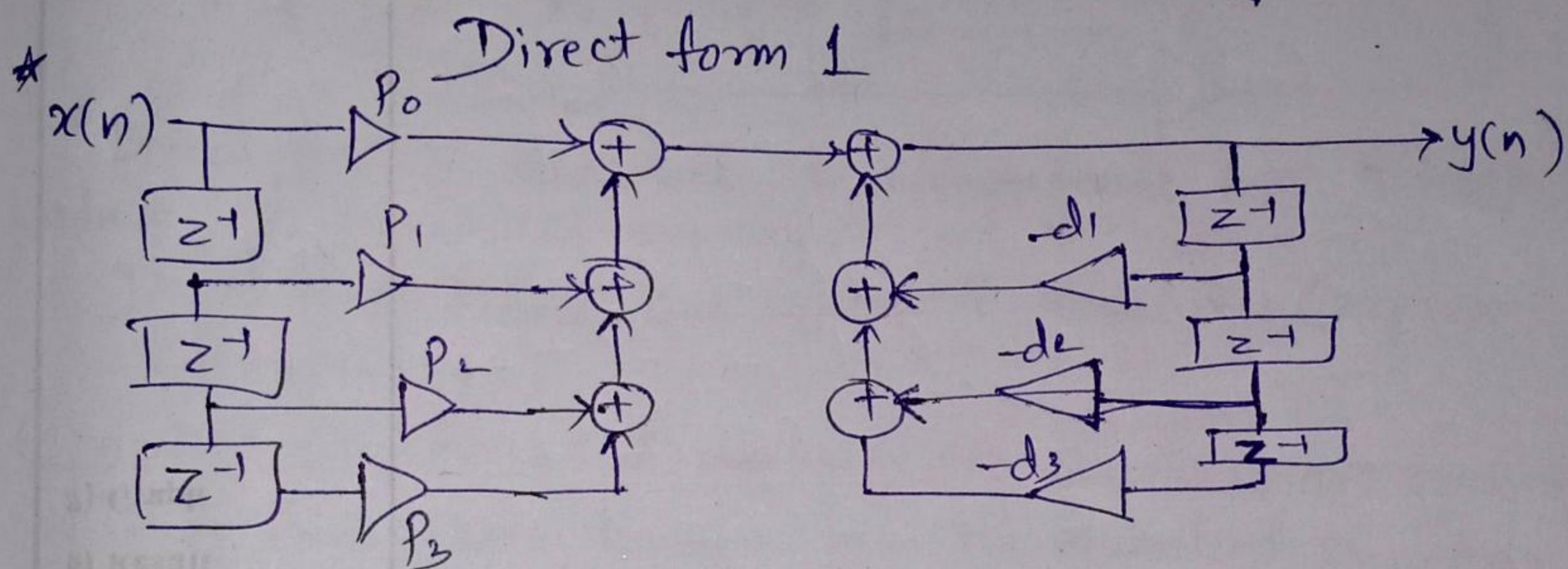
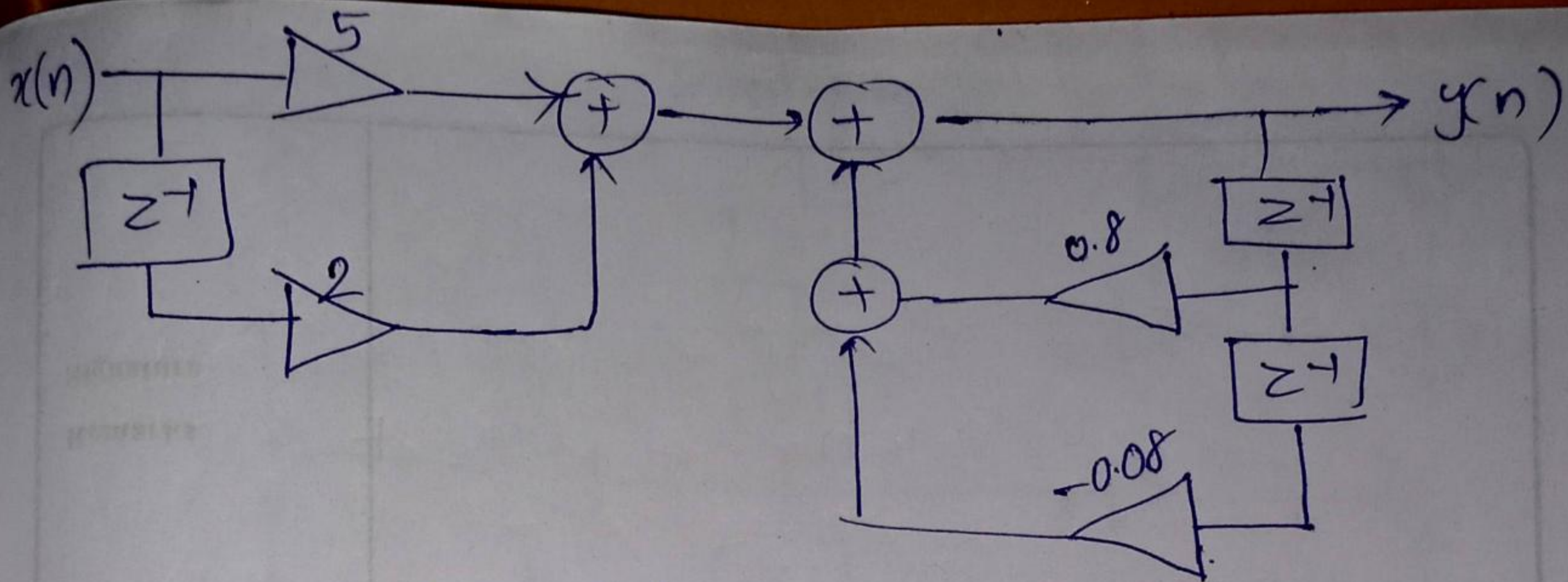
$$W(z) = Y(z) [1-0.8z^{-1}+0.08z^{-2}]$$

$$W(z) = Y(z) - 0.8z^{-1}Y(z) + Y(z)0.08z^{-2}$$

$$Y(z) = W(z) + 0.8z^{-1}Y(z) - 0.08z^{-2}Y(z)$$

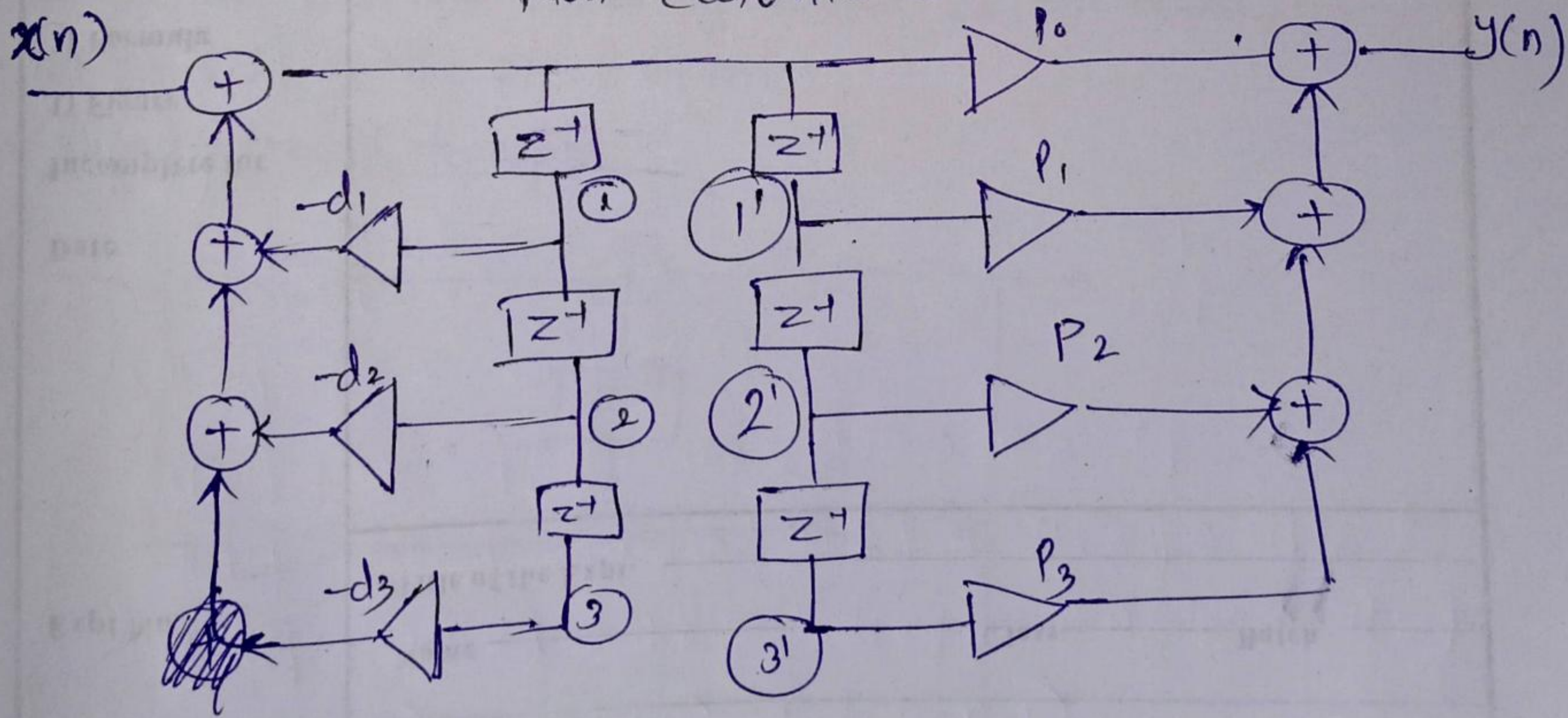
$$w(n) = 5x(n) + 2x(n-1)$$

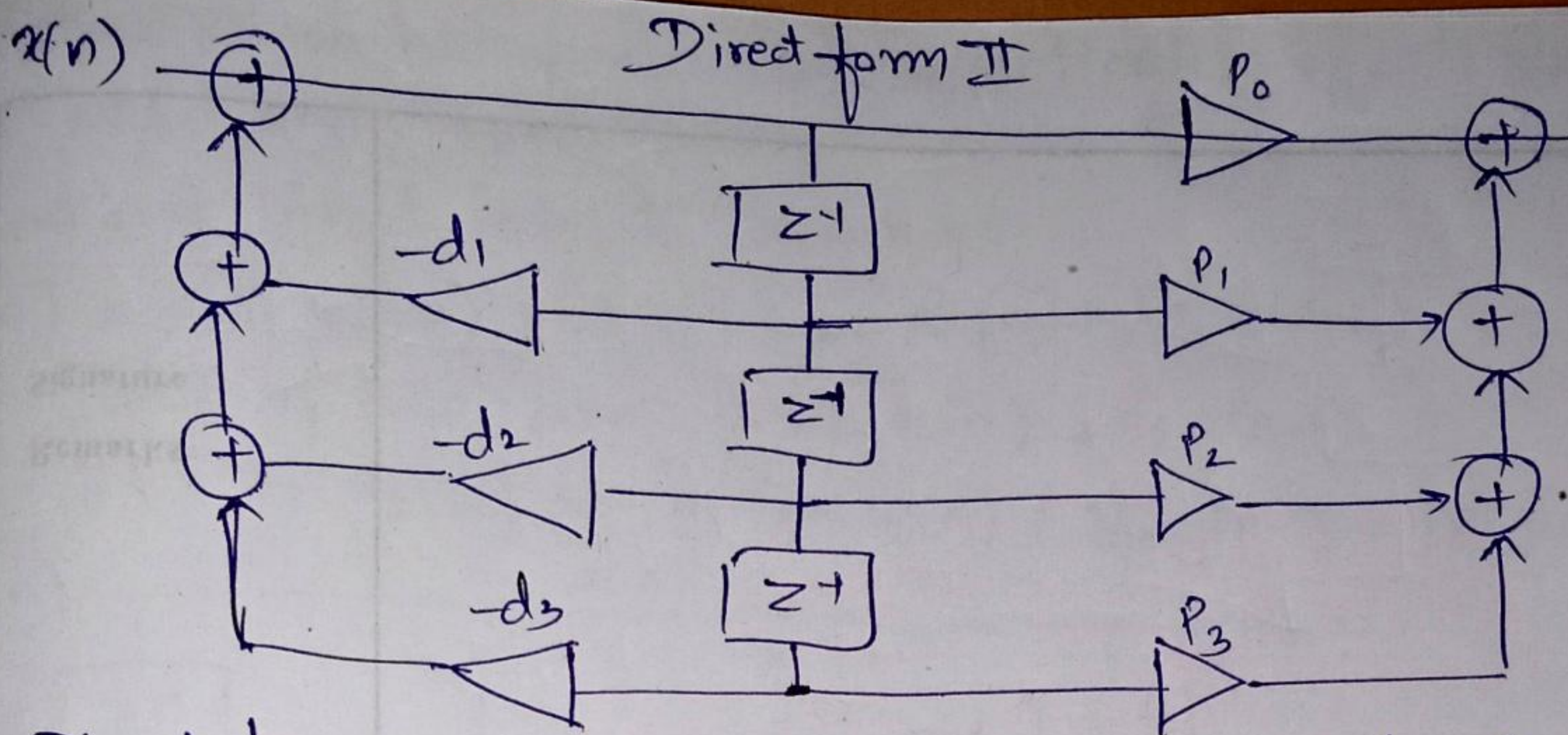
$$y(n) = w(n) + 0.8y(n-1) - 0.08y(n-2)$$



Since LTI systems are linear systems, we can interchange the positions of $H_1(z)$ & $H_2(z)$

Non canonic structure.





Direct form II structure is advantageous bcoz it reduces hardware & minimizes memory.

Direct form I & II are equivalent structures. (represents same difference eqⁿ).

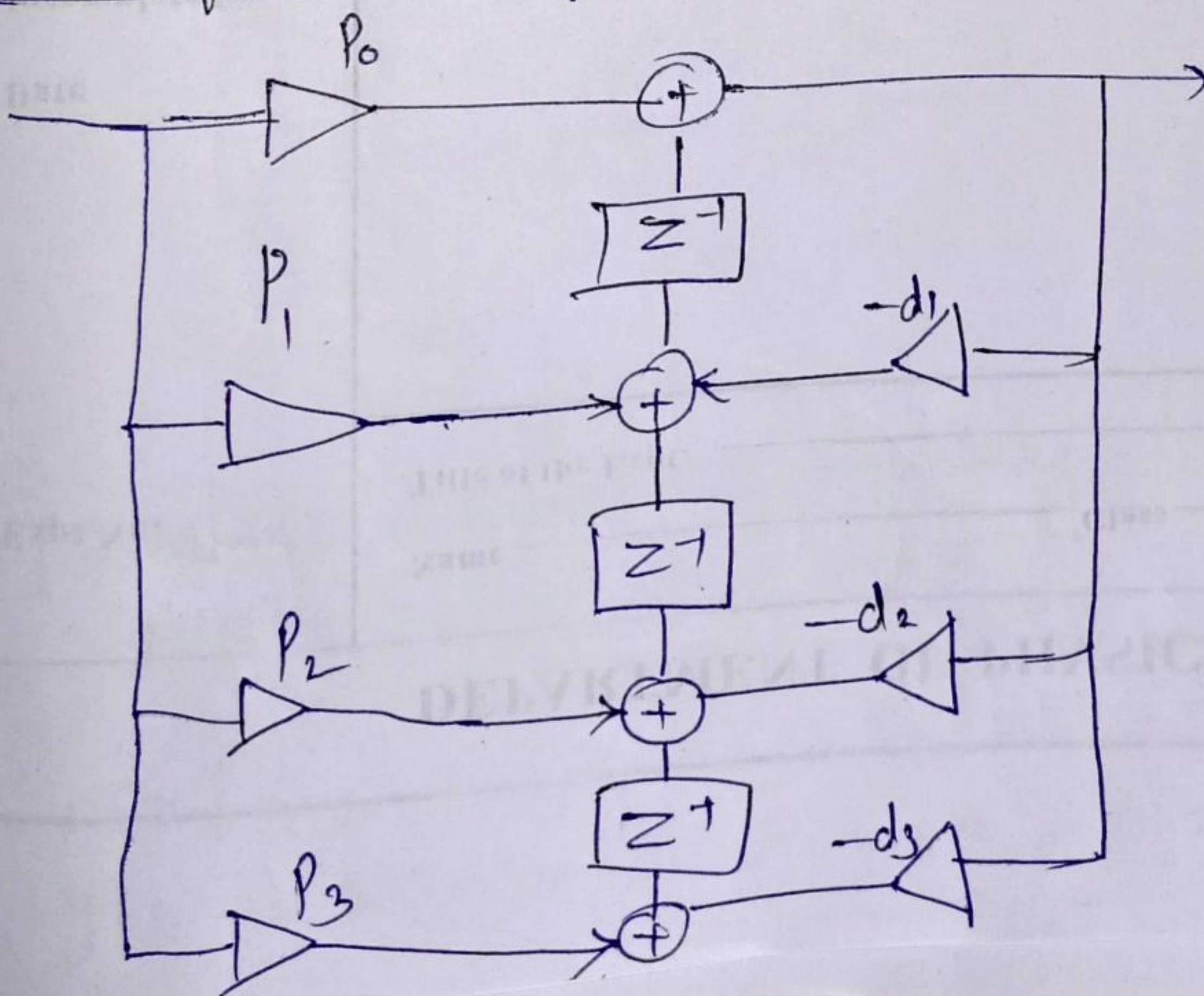
Equivalent Structures: 2 digital filter structures are equivalent if they have same transfer fn. The equivalent structure is obtained by obtaining transpose operation.

1) Reverse all paths.

2) Replace pick off nodes by adders & vice versa.

3) Interchange the I/P & o/p nodes.

* Direct form II transpose



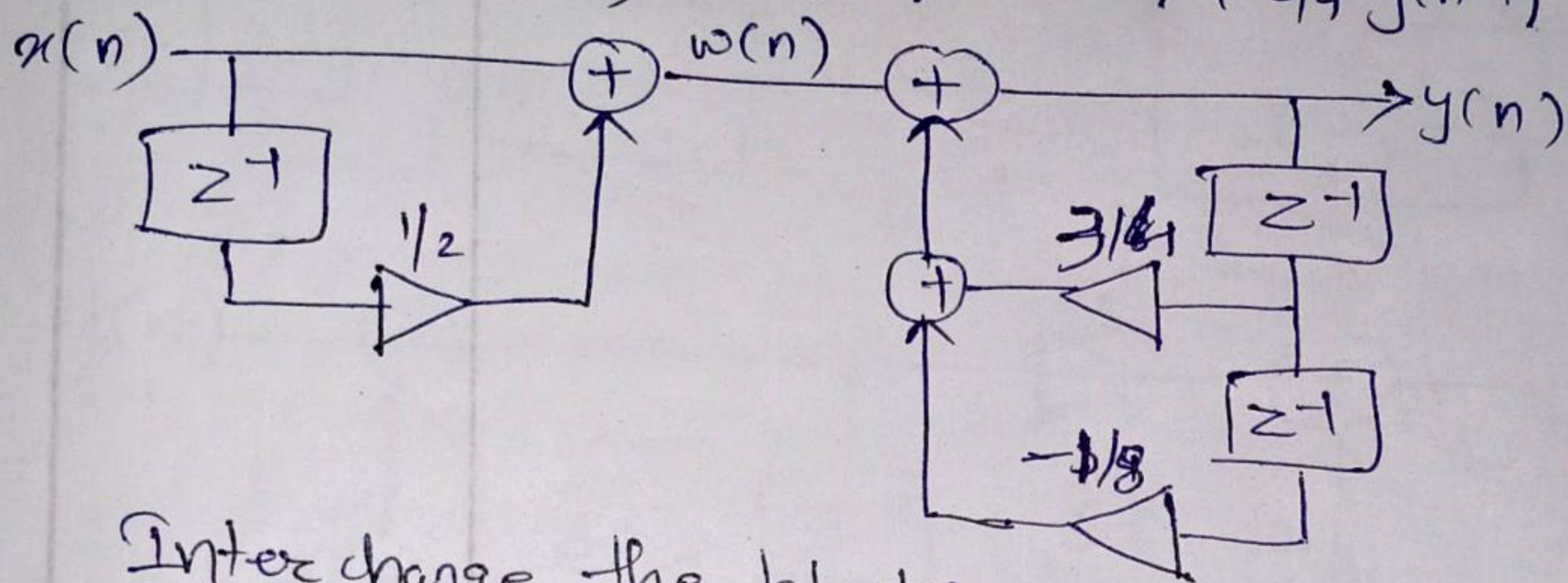
Structure which has finite least quantization effects from finite word length implementation.

Q) Construct Direct form II structure

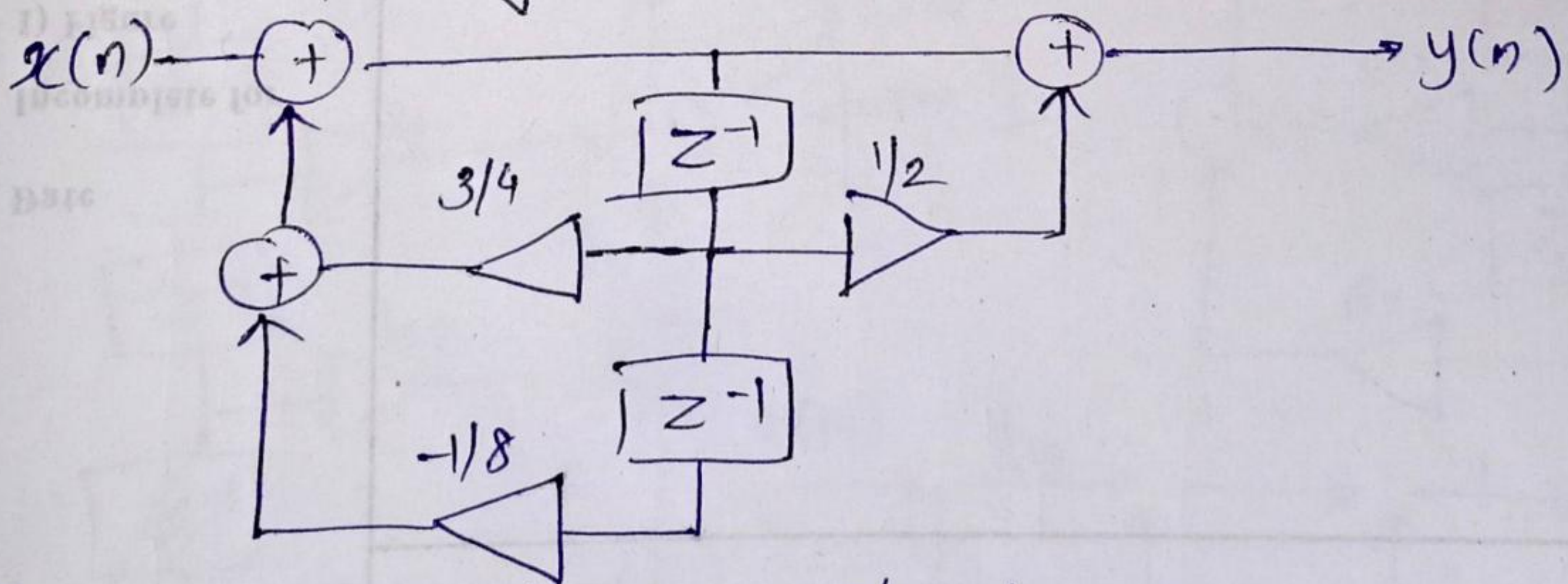
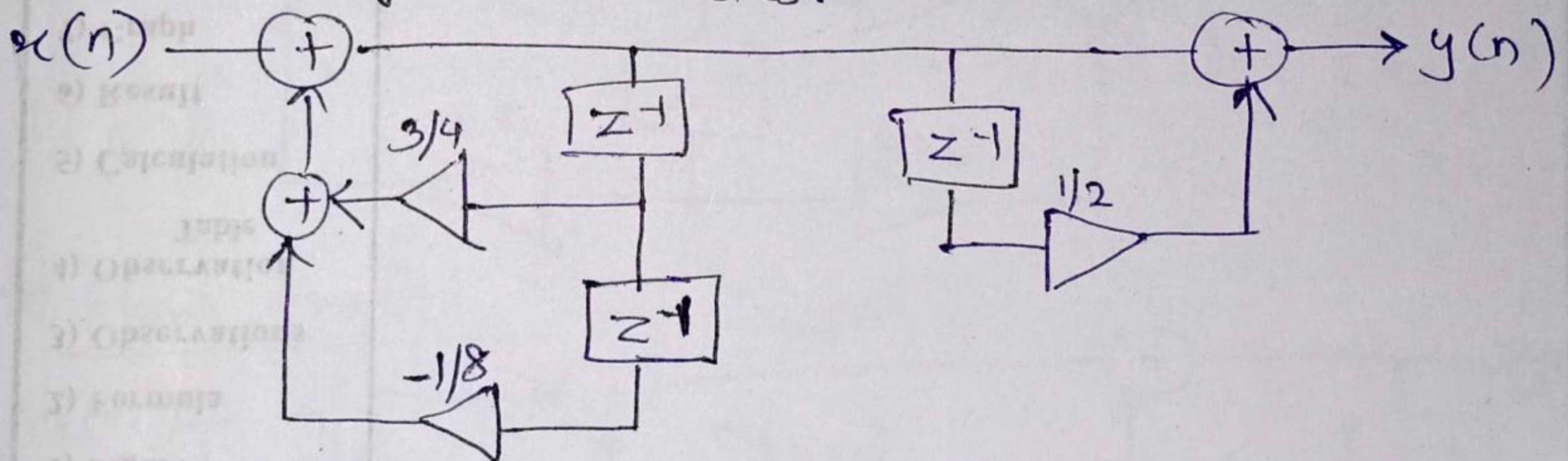
$$y(n) = -3/4 y(n-1) + 1/8 y(n-2) = x(n) + 1/2 x(n-1)$$

$$\rightarrow \text{IzT of } H_1(z) \Rightarrow w(n) = x(n) + 1/2 x(n-1)$$

$$\text{IzT of } H_2(z) \Rightarrow y(n) = w(n) + 3/4 y(n-1) - 1/8 y(n-2)$$



Interchange the blocks.



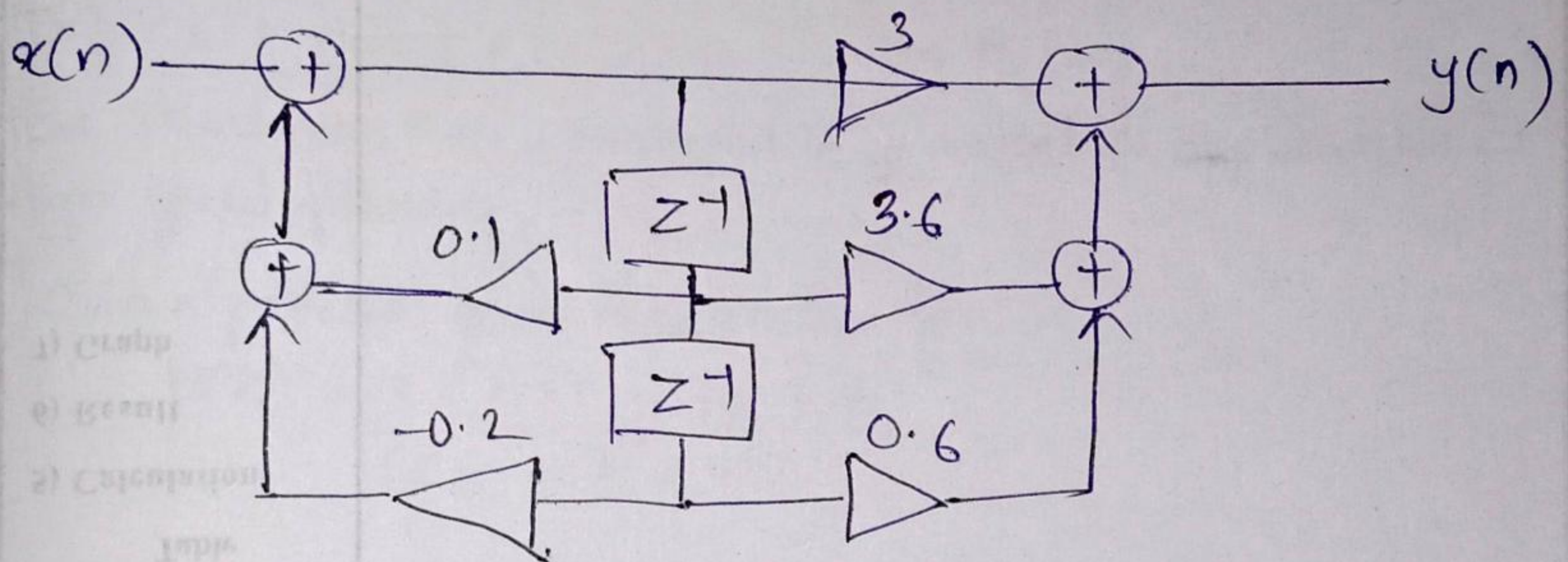
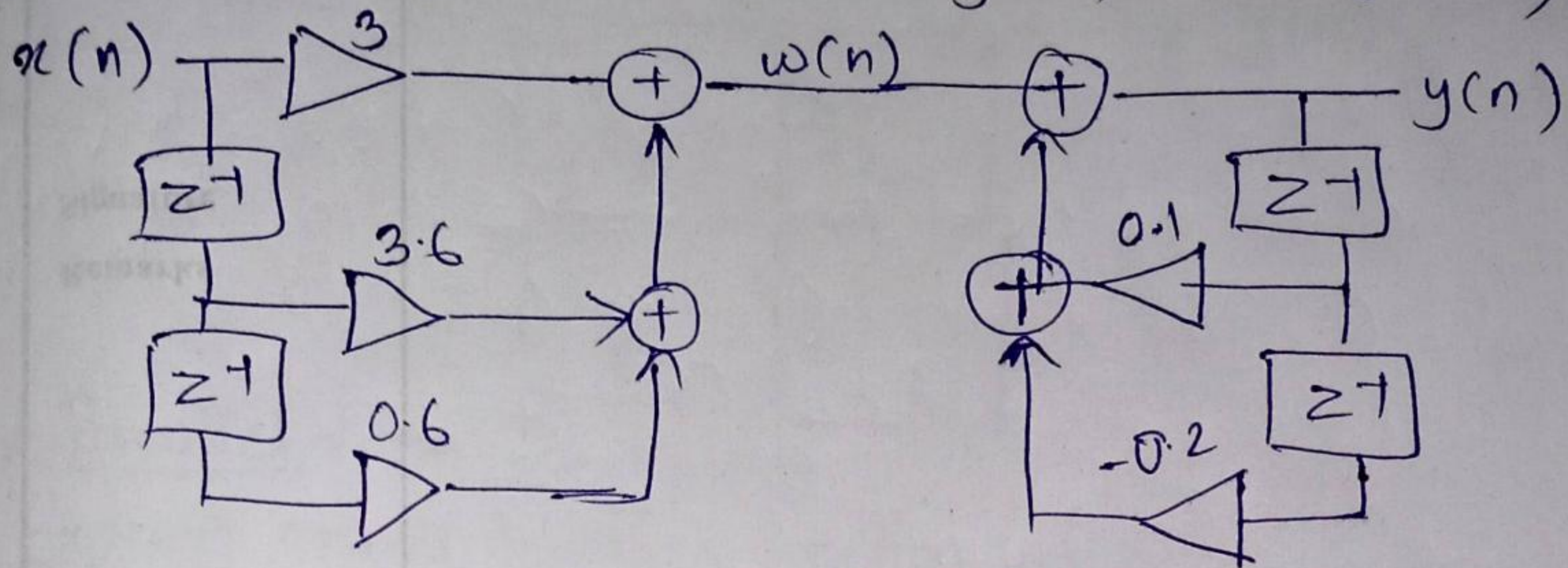
Q.] Obtain Direct form II structure

$$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

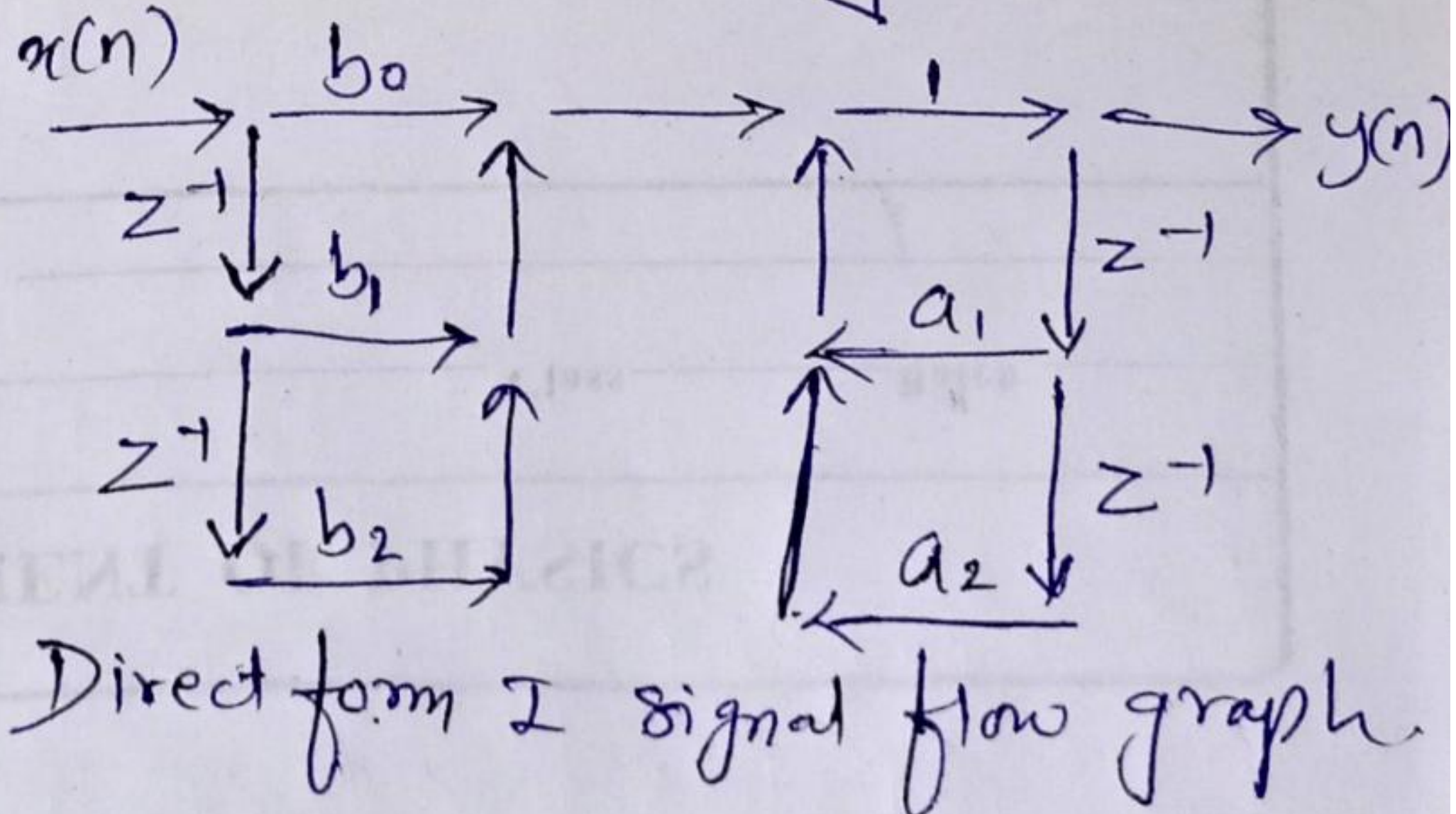
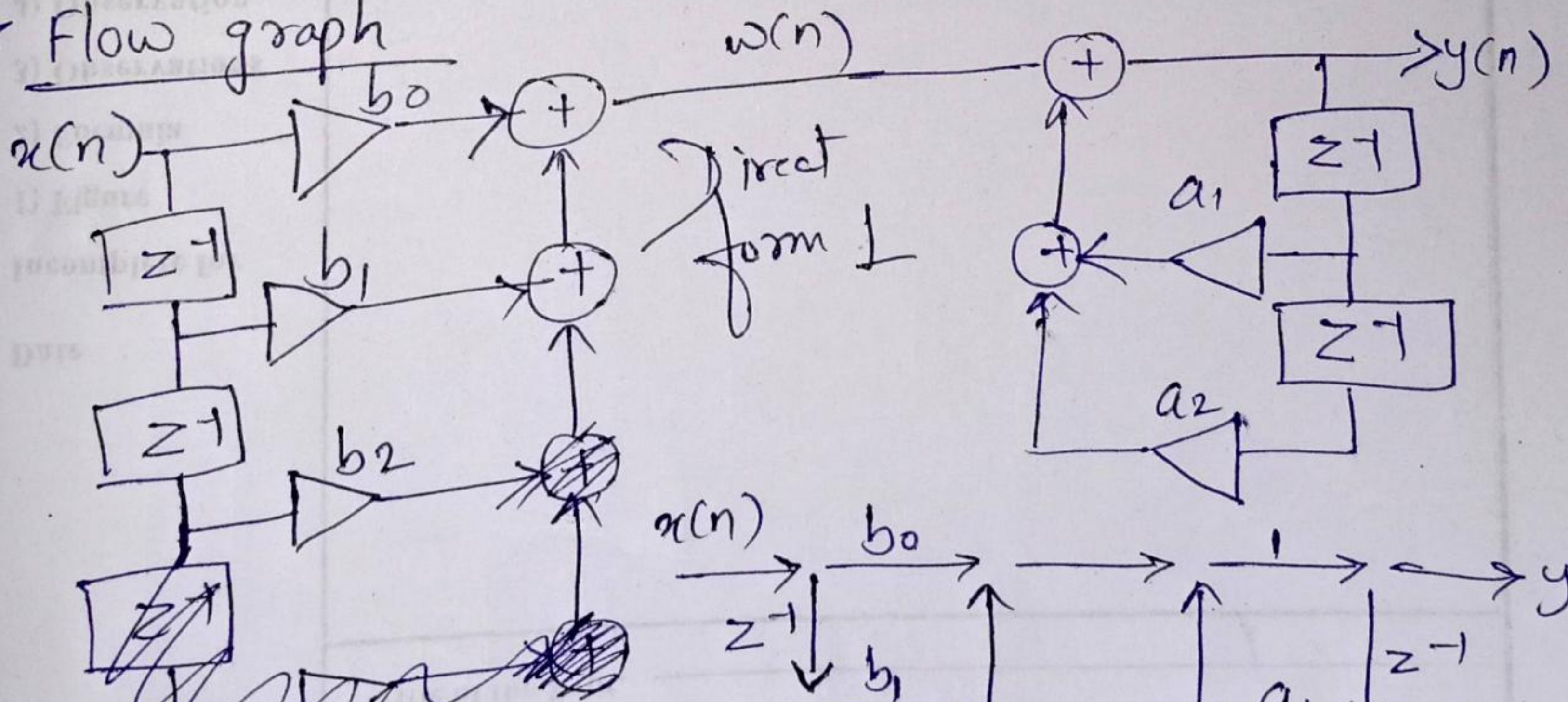
$$w[z] = 3x[z] + 3.6x[z]z^{-1} + 0.6x[z]z^{-2}$$

$$y[z] = w[z] + 0.1y[z]z^{-1} - 0.2y[z]z^{-2}$$

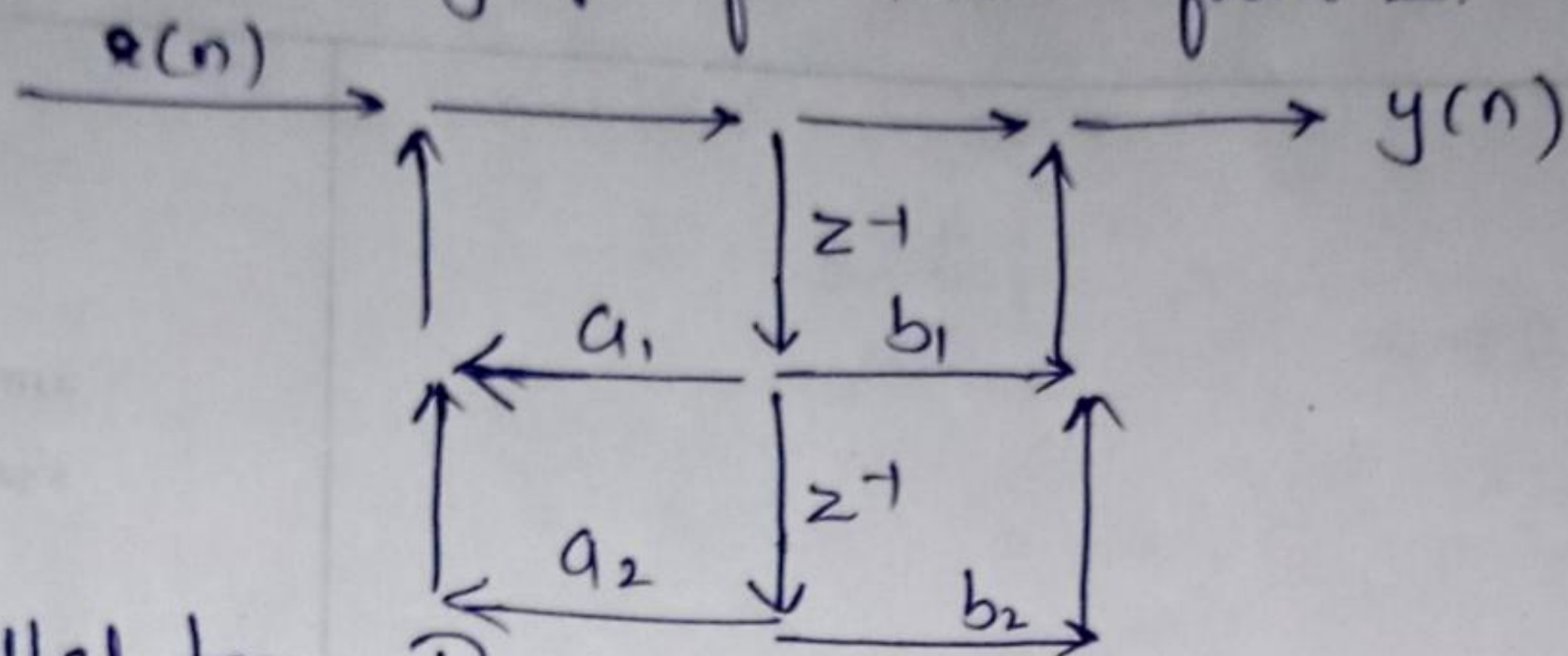
$\text{IZT of } H_1(z) \quad w(n) = 3x(n) + 3.6x(n-1) + 0.6x(n-2)$
 $\text{IZT of } H_2(z) \quad y(n) = w(n) + 0.1y(n-1) - 0.2y(n-2)$



★ Flow graph



Signal flow graph for Direct form 2.



* Parallel form Realization.

Express overall transfer function as,

$$H(z) = C + H_1(z) + H_2(z) + \dots + H_K(z) \dots \quad C = \text{constant.}$$

Here structures are connected in parallel & application $\rightarrow$ High speed filtering.

8.] Obtain parallel form Realization for.

$$H(z) = \frac{3(2z^2 + 5z + 4)}{(2z + 1)(z + 2)}$$

$$\rightarrow H(z) = \frac{\frac{1}{2}(6z^2 + 15z + 12)}{(z + 1/2)(z + 2)}$$

$$\frac{H(z)}{z} = \frac{3/2(2z^2 + 5z + 4)}{z(z + 1/2)(z + 2)}$$

$$= \frac{A_1}{z} + \frac{A_2}{z + 1/2} + \frac{A_3}{z + 2}$$

$$N = A_1(z + 1/2)(z + 2) + A_2(z)(z + 2) + A_3(z + 1/2)z$$

$$= A_1(z^2 + 2z + z/2 + 1) + A_2(z^2 + 2z) + A_3(z^2 + z/2)$$

$$= \underline{A_1} z^2 + \underline{\frac{5}{2} z A_1} + \underline{A_1} + \underline{A_2} z^2 + \underline{2A_2} z + \underline{A_3} z^2 + \underline{\frac{A_3}{2} z}$$

$$A_1 + A_2 + A_3 = 3, \quad \frac{5}{2} A_1 + 2A_2 + \frac{A_3}{2} = \frac{15}{2}$$

$$\boxed{A_2 = -4} \quad \boxed{A_1 = 6}$$

$$1.5 A_3 = 15 - 13.5$$

$$\boxed{A_3 = 1}$$

$$15 + 2A_2 + \frac{A_3}{2} = \frac{15}{2}$$

$$2A_2 + 2A_3 = -13.5$$

$$15 - 1.5 A_3 = 13.5$$

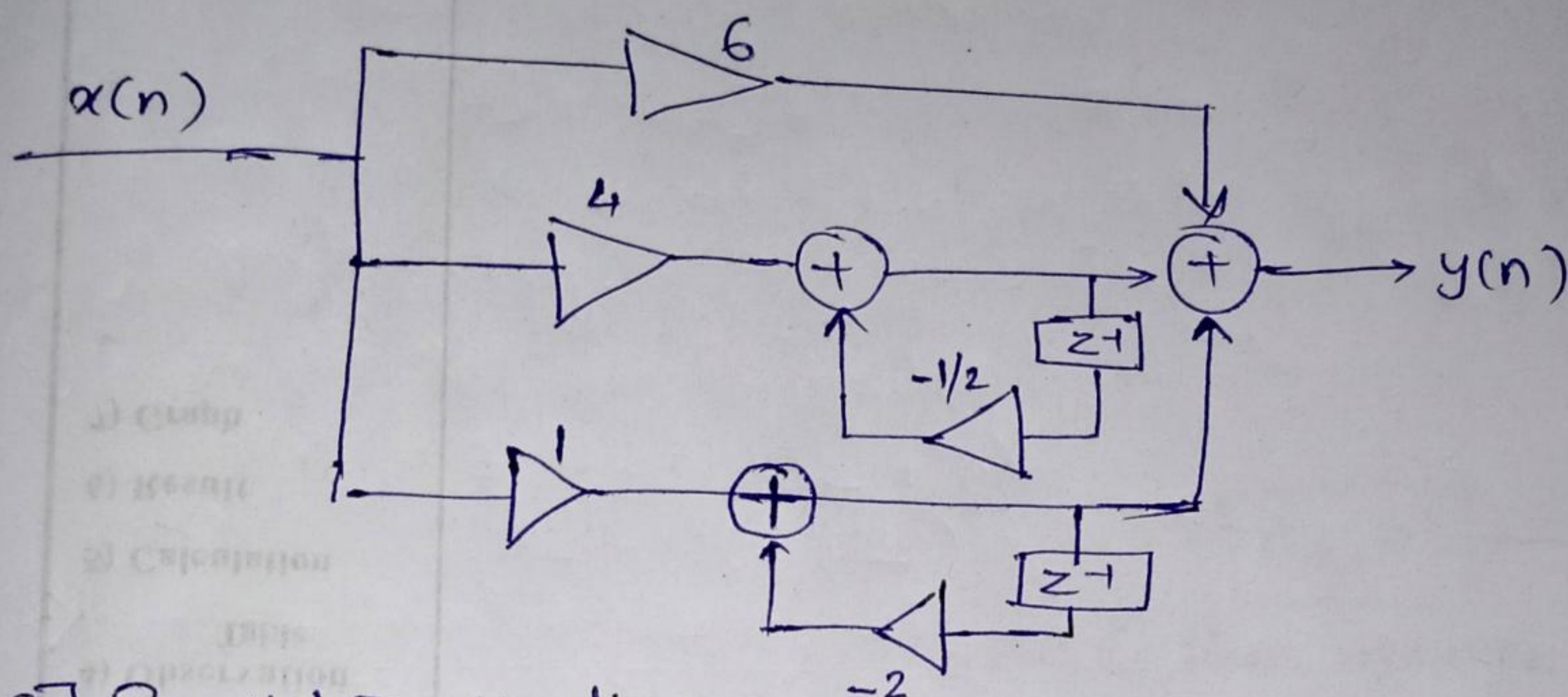
$$0.3 A_1 - 0.5 + 0.5 A_1 = -1$$

$$\frac{H(z)}{z} = \frac{6}{z} - \frac{4}{z+1/2} + \frac{1}{z+2}$$

$$H(z) = \frac{6}{z} \times z - \frac{4z}{z+1/2} + \frac{z}{z+2}$$

$$H(z) = 6 - \frac{4z}{z+1/2} + \frac{z}{z+2}$$

$$H(z) = 6 - \frac{4}{1+1/2 z^{-1}} + \frac{1}{1+2z^{-1}}$$



Q.] Parallel Realization

$$H(z) = \frac{1-z^{-1}}{1-0.2z^{-1}-0.15z^{-2}}$$

First convert this eqⁿ into +ve powers of z

$$\rightarrow H(z) = \frac{z^2 + z}{z^2 - 0.2z - 0.15}$$

$$= \frac{z(z-1)}{(z-0.5)(z+0.3)}$$

$$= \frac{A_1 z + B_1}{z-0.5} + \frac{A_2}{z+0.3} = \frac{(z-1)}{(z-0.5)(z+0.3)}$$

$$= \frac{A_1}{z-0.5} + \frac{A_2}{z+0.3}$$

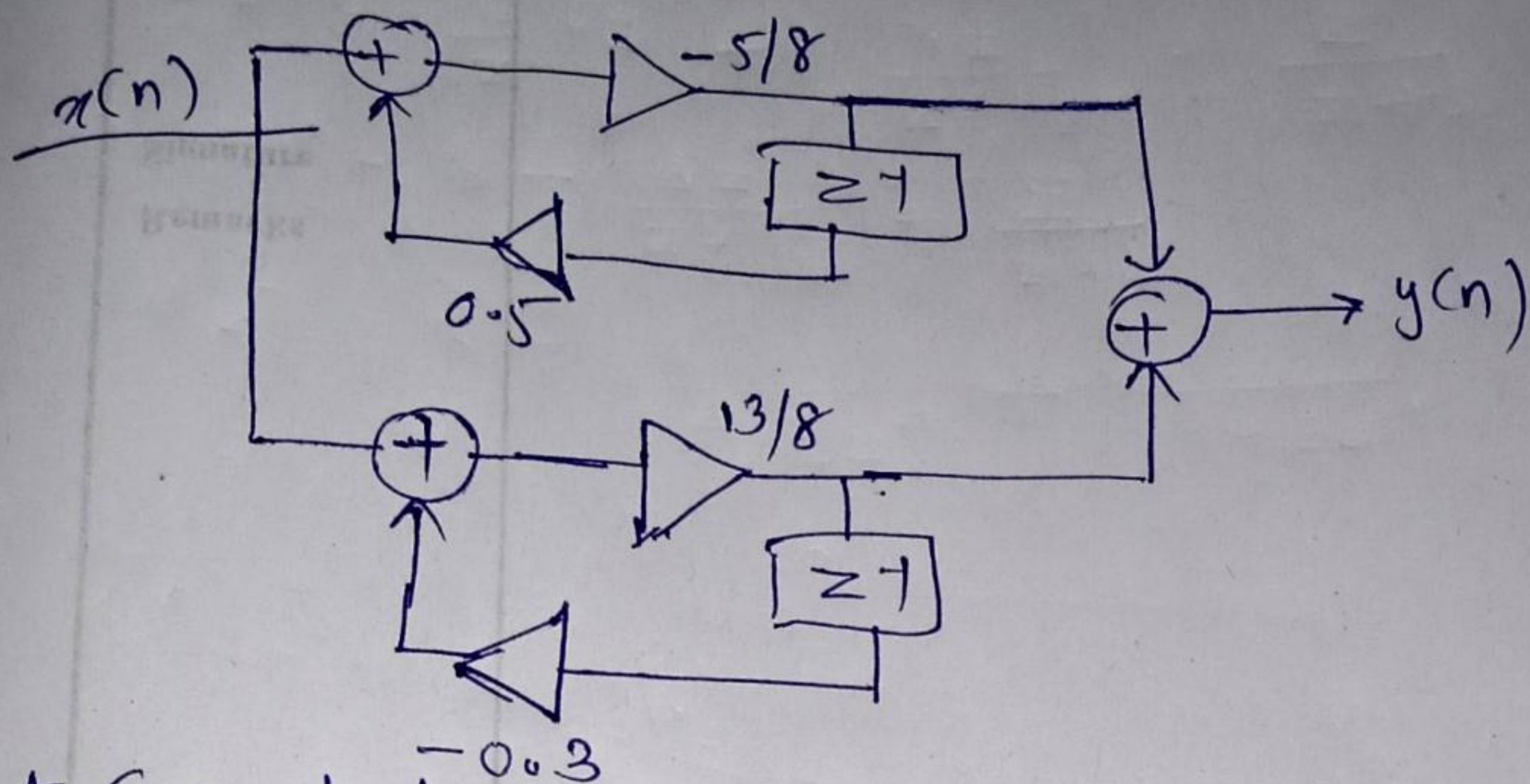
$$N = A_1 z + 0.3 A_1 + A_2 z - 0.5 A_2$$

$$= (A_1 + A_2)z + 0.3 A_1 - 0.5 A_2$$

$$A_1 + A_2 = 1 \quad 0.3 A_1 - 0.5 A_2 = -1$$

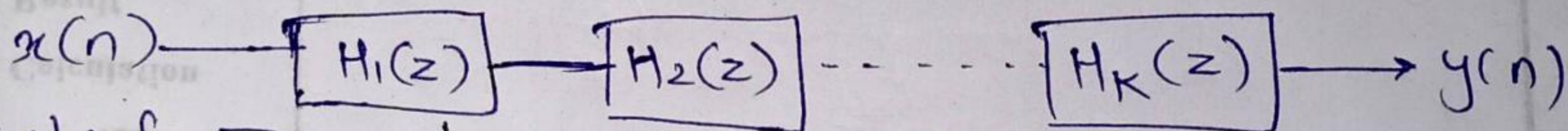
$$\therefore \boxed{A_1 = -5/8} \quad \boxed{A_2 = 13/8} \quad \begin{aligned} 0.3 A_1 - 0.5(1-A_1) &= -1 \\ 0.3 A_1 - 0.5 + 0.5 A_1 &= -1 \end{aligned}$$

$$\frac{H(z)}{z} = \frac{-5/8}{z-0.5} + \frac{13/8}{z+0.3}$$



★ Cascade form Realization

$$H(z) = H_1(z) \cdot H_2(z) \cdot \dots$$



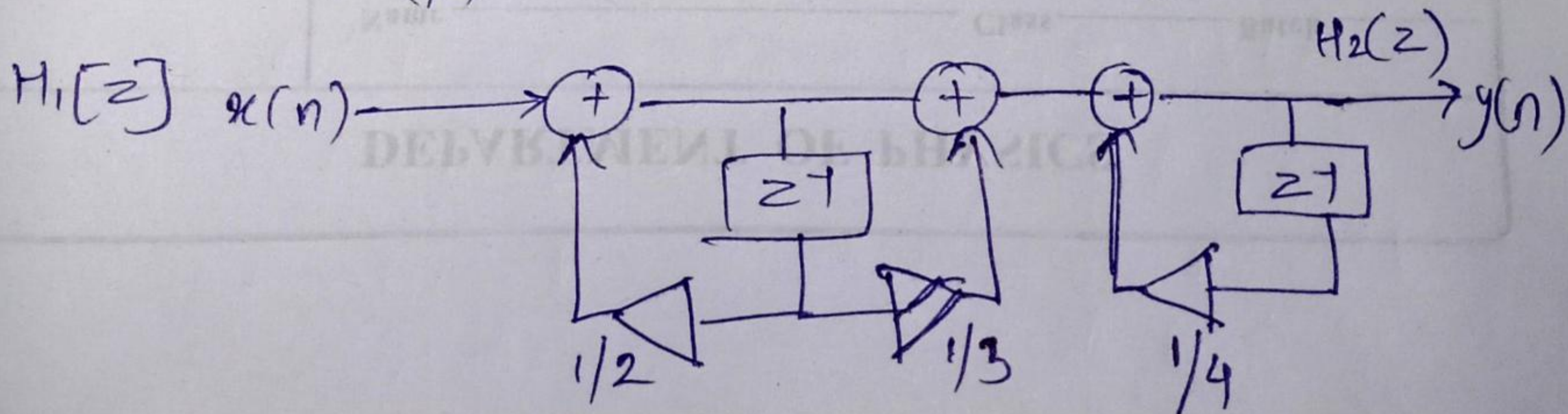
first go for Direct form 2 realization & then arrange.

Q. $y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{3}x(n-1)$
Represent in Cascade form.

$$\begin{aligned} \underline{H(z)} &= \frac{Y(z)}{X(z)} = \frac{1 + (1/3)z^{-1}}{1 - (3/4)z^{-1} + (1/8)z^{-2}} \\ &= \frac{1 + (1/3)z^{-1}}{(1 - (1/2)z^{-1})(1 - (1/4)z^{-1})} \end{aligned}$$

$$H_1(z) = \frac{1 + (1/3)z^{-1}}{1 - (1/2)z^{-1}}$$

$$H_2(z) = \frac{1}{1 - (1/4)z^{-1}}$$



Module 3 FIR filter.

* Design of a simple filter. (High pass)

I/p signal = sum of 2 cosine sequences.

0.1 rad/sample, 0.4 rad/sample.

1st seq. freq.

2nd seq. freq.

For simplicity we assume that filter is FIR, length = 3 with impulse response

$$h[0] = h[2] = \alpha_0, \quad h[1] = \alpha_1$$

$$y(n) = \sum_{k=0}^{n-1} h(k) x(n-k)$$

The digital filtering is performed using difference eqⁿ

$$y[n] = h[0] x[n] + h[1] x[n-1] + h[2] x[n-2]$$

$$y[n] = \alpha_0 x[n] + \alpha_1 x[n-1] + \alpha_0 x[n-2] \quad \text{--- (1)}$$

$$y[n] = \text{o/p}, \quad x[n] = \text{i/p}.$$

Choose suitable parameters for filter.

they are α_0, α_1 ,

Since we are designing HPF, we want the system to block 0.1 rad/sample & pass 0.4 rad/sample.

This is our design objective.

The frequency response of FIR filter is given as,

$$H(e^{j\omega}) = \sum_{k=0}^{N-1} h[k] e^{-j\omega k} \quad k=0, 1, 2.$$

$$H(e^{j\omega}) = h[0] + h[1] e^{-j\omega} + h[2] e^{-2j\omega}$$

$$= h[0] [1 + e^{-2j\omega}] + h[1] e^{-j\omega}$$

$$H(e^{j\omega}) = \alpha_0 [1 + e^{-2j\omega}] + \alpha_1 e^{-j\omega}$$

Multiply & divide $[1 + e^{-2j\omega}]$ by $2e^{-j\omega}$

$$\therefore H(e^{j\omega}) = 2\alpha_0 \left[\frac{e^{j\omega} + e^{-j\omega}}{2} \right] e^{-j\omega} + \alpha_1 e^{-j\omega}$$

$$= [2\alpha_0 \cos \omega] e^{-j\omega} + \alpha_1 e^{-j\omega}$$

$$H(e^{j\omega}) = e^{-j\omega} [2\alpha_0 \cos \omega + \alpha_1]$$

The magnitude & phase functions of this filter is

$$|H(e^{j\omega})| = |2\alpha_0 \cos \omega + \alpha_1|$$

$\beta = 0$ when $2\alpha_0 \cos \omega + \alpha_1$ is greater than 0,
 $\beta = \pi$ when $2\alpha_0 \cos \omega + \alpha_1 < 0$

$$\theta(\omega) = -\omega + \beta$$

ω = angular frequency containing 0.1 rad/sample & 0.4 rad/sample.

$$\therefore 2\alpha_0 \cos 0.1 + \alpha_1 = 0 \quad \& \quad \text{Solving these 2 eq's simultaneously,}$$

$$2\alpha_0 \cos 0.4 + \alpha_1 = 1$$

$$\alpha_0 = -6.76195, \quad \alpha_1 = 13.456335$$

Substituting these values in eqⁿ ①

$$y[n] = -6.76195 [x[n] + x[n-2]] + 13.456335 x[n-1]$$

with $x[n] = \{\cos(0.1n) + \cos(0.4n)\} u(n)$

I/p = causal sequence.

* Basic approaches to FIR filter design.

A causal FIR filter transfer fn.

$$H(z) = \sum_{n=0}^N h[n] z^{-n} \quad \& \quad \text{corresponding frequency}$$

response is given as $H(e^{j\omega}) = \sum_{n=0}^N h[n] e^{-j\omega n}$

Accomplished by finding

1> Finding either the impulse response $h(n)$ or

2> (N+1) samples of its frequency response $H(e^{j\omega})$.

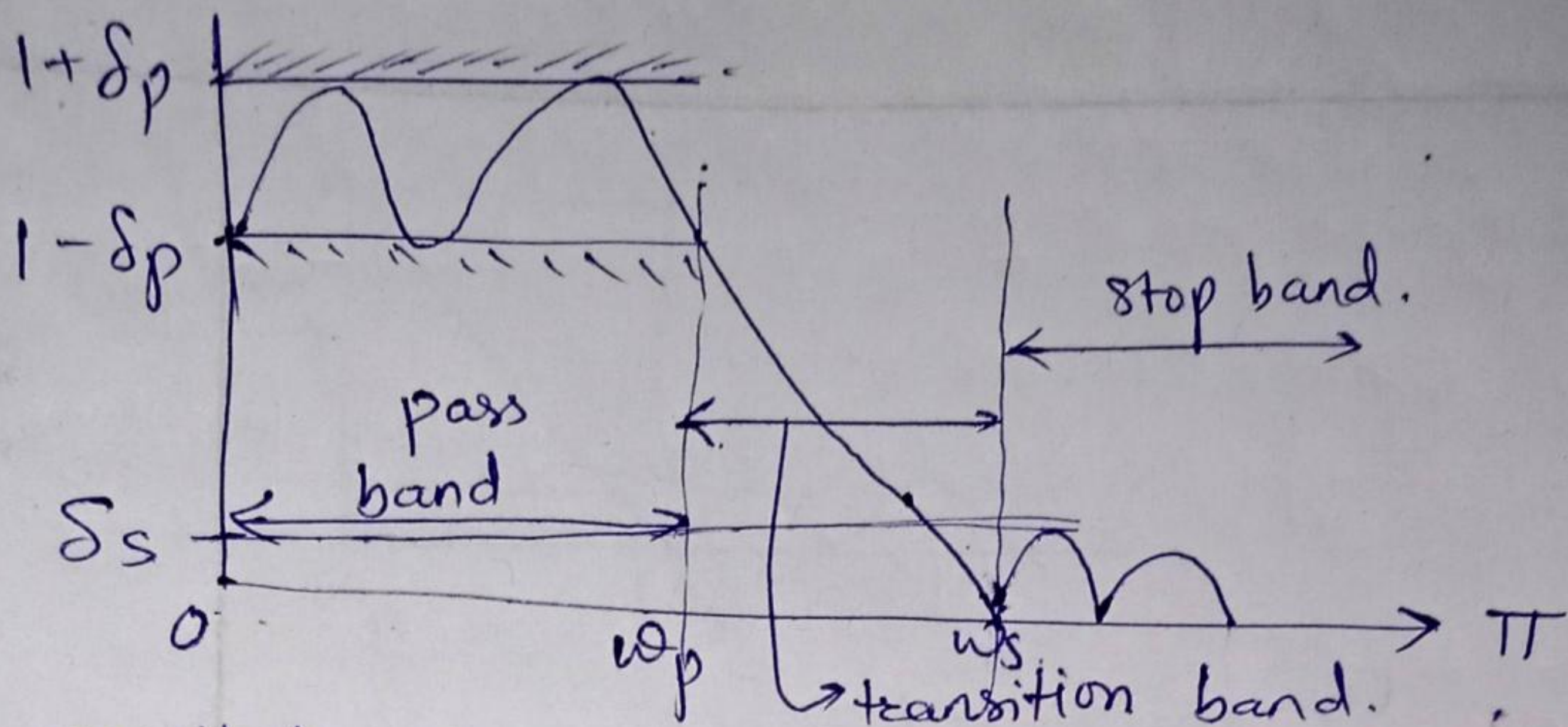
Linear phase design is also required which is accomplished

by $\rightarrow h[n] = \pm h[N-n]$

* Direct approaches to design FIR filter.

1> Windowed Fourier series. 2> Frequency sampling method

Typical magnitude specifications for digital low pass filter



magnitude approximates unity $\rightarrow$ pass band.
with error of $\pm \delta_p$

For ideal, transition band should be as small as possible.

Pass band $\rightarrow$ It is defined by ~~zero~~ $0 \leq \omega \leq \omega_p$.

magnitude here approximates unity. in 0 to ω_p . & with an error $\pm \delta_p$. $\rightarrow 1 - \delta_p \leq |G(e^{j\omega})| \leq 1 + \delta_p$ $\rightarrow |\omega| \leq \omega_p$

Stop band $\rightarrow$ defined by $\omega_s \leq \omega \leq \pi$

magnitude approximately 0 with error δ_s .

$\rightarrow \delta_s \leq |G(e^{j\omega})| \leq \delta_s$ $\omega_s \leq \omega \leq \pi$.

ω_p = pass band edge frequency.

ω_s = stopband edge frequency.

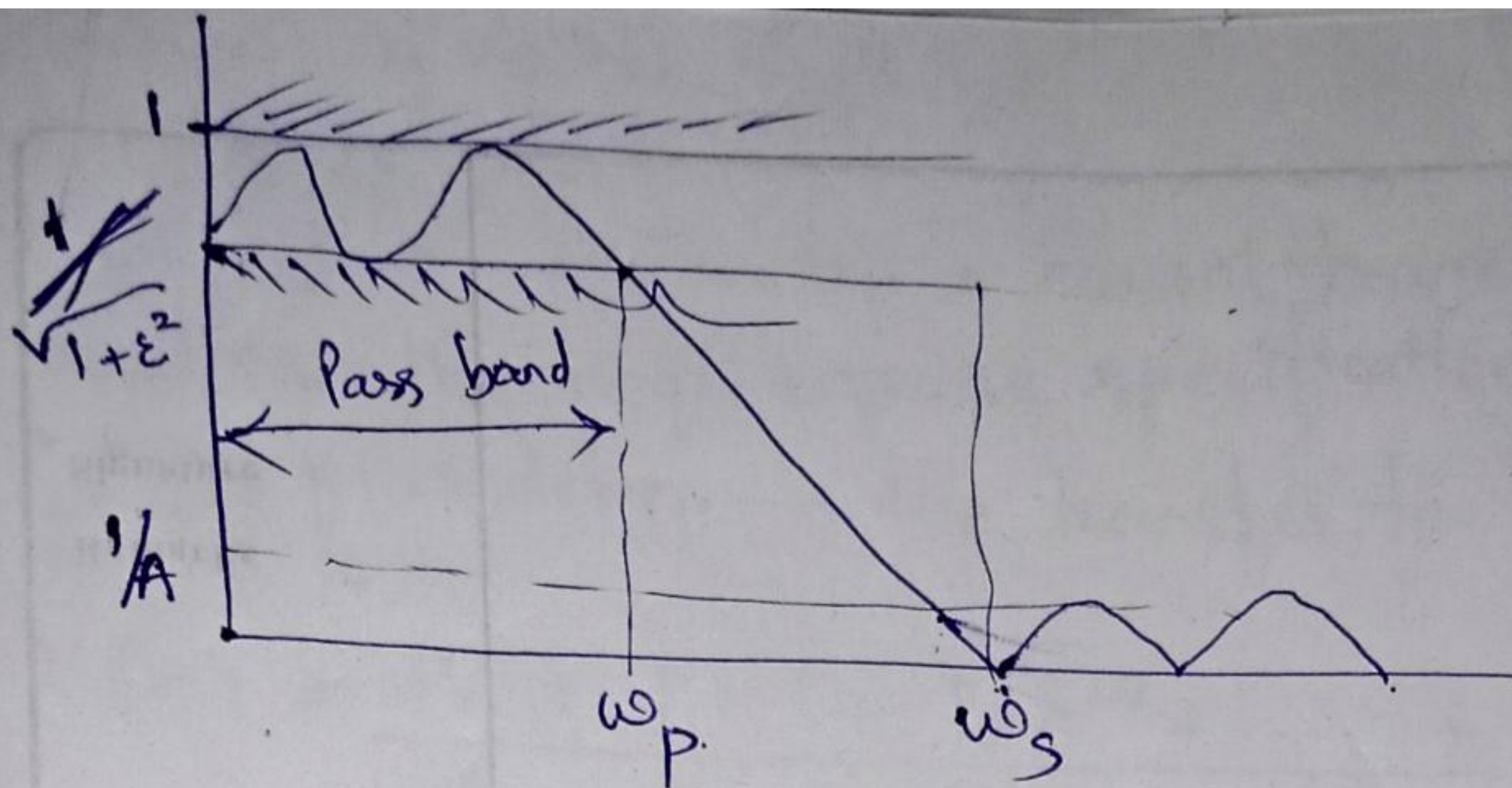
δ_p, δ_s = peak ripple values.

* Digital filter specifications $\rightarrow$ loss function.
 $A(\omega) = -20 \log_{10} |G(e^{j\omega})|$ in dB.

* Peak passband ripple
 $\alpha_p = -20 \log_{10} (1 - \delta_p)$ in dB.

* Minimum stopband attenuation
 $\alpha_s = -20 \log_{10} \delta_s$ in dB

* In case of analog filter, magnitude response specifications



$\frac{1}{\sqrt{1+\epsilon^2}}$ = minimum value of magnitude in pass band

maximum stopband magnitude is $1/A$.

maximum passband attenuation is $\alpha_{\max}$

$$\alpha_{\max} = 20 \log_{10} \left(\sqrt{1+\epsilon^2} \right) \text{ in dB.}$$

Let f_T = Sampling frequency in Hz.

f_p = Pass band edge frequency. in Hz.

f_s = Stop band edge frequency. in Hz

Angular freq $\omega_s, \omega_p \rightarrow$ radians.

$$\omega_p = \frac{\Omega_p}{f_T} = \frac{2\pi f_p}{f_T} = 2\pi f_p T$$

$$1/f_T = T$$

$$\omega_s = \frac{\Omega_s}{f_T} = \frac{2\pi f_s}{f_T} = 2\pi f_s T$$

Q] Specified pass band & stop band edge frequencies of digital high pass filter operating at sampling rate 25 kHz be 7 kHz & 3 kHz respectively. Determine corresponding normalized angular band edge frequencies.

$$\rightarrow f_T = 25 \text{ kHz}, f_p = 7 \text{ kHz}, f_s = 3 \text{ kHz}, \omega_s = ?, \omega_p = ?$$

$$\omega_s = \frac{2\pi f_s}{f_T} = \frac{6}{25} \pi = 0.75 \pi$$

$$\omega_p = \frac{2\pi f_p}{f_T} = \frac{14}{25} \pi = 1.75 \pi$$

* Selection of filter type.

→ FIR or IIR.

→ Objective → develop a causal transfer fn. $H(z)$, meeting the freq. response specifications.

→ IIR filter design → the transfer fn. is real rational fn. of z^{-1}

$$H[z] = \frac{P_0 + P_1 z^{-1} + P_2 z^{-2} + \dots + P_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \dots + d_N z^{-N}}$$

$H(z)$ must be stable transfer fn. for reduced computational complexity. N must be as small as possible.

→ FIR filter design → transfer fn. is polynomial in z^{-1}
 $H[z] = \sum_{n=0}^N h[n] z^{-n}$ → Here also N should be small.

For linear phase → $h[n] = \pm h[N-n]$

* Estimated value → filter order

- For reduced computational complexity, the filter order should be the smallest integer greater than or equal to the estimated value.

* Order estimation (FIR filter)

Kaiser's formula (gives good estimation)

$$N = \frac{-20 \log_{10}(\sqrt{\delta_p \delta_s}) - 13}{14.6 (\omega_s - \omega_p) / 2\pi}$$

Estimate the order of linear phase FIR filter with following specification →

pass band edge $F_p = 1.8 \text{ KHz}$

Stop band edge $F_s = 2 \text{ KHz}$

peak passband ripple $\alpha_p = 0.1 \text{ dB}$

minimum stopband attenuation $\alpha_s = 35 \text{ dB}$

Sampling rate = 12 KHz

$$\alpha_p = -20 \log_{10}(1 - \delta_p) \quad , \quad \alpha_s = -20 \log_{10}(\delta_s)$$

Rearranging this,

$$\delta_p = 1 - 10^{-\alpha_p/20}$$

$$\delta_p = 0.0114469$$

$$\delta_s = 10^{-\alpha_s/20}$$

$$\delta_s = 0.0177828$$

$$\therefore N = \frac{-20 \log_{10}(0.0142673) - 13}{14.6 (2000 - 1800) / 12000} = \frac{23.9131}{0.24333}$$

$$= 98.2730$$

N should be integer

$$\therefore \boxed{N = 99}$$

* Bellanger's Formula

$$N \approx \frac{2 \log_{10}(10 \delta_p \delta_s)}{3(\omega_s - \omega_p) / 2\pi} - 1$$

Estimation of filter order N of the FIR filter is inversely proportional to transition bandwidth $(\omega_s - \omega_p)$. It does not depend upon the actual location of transition band.

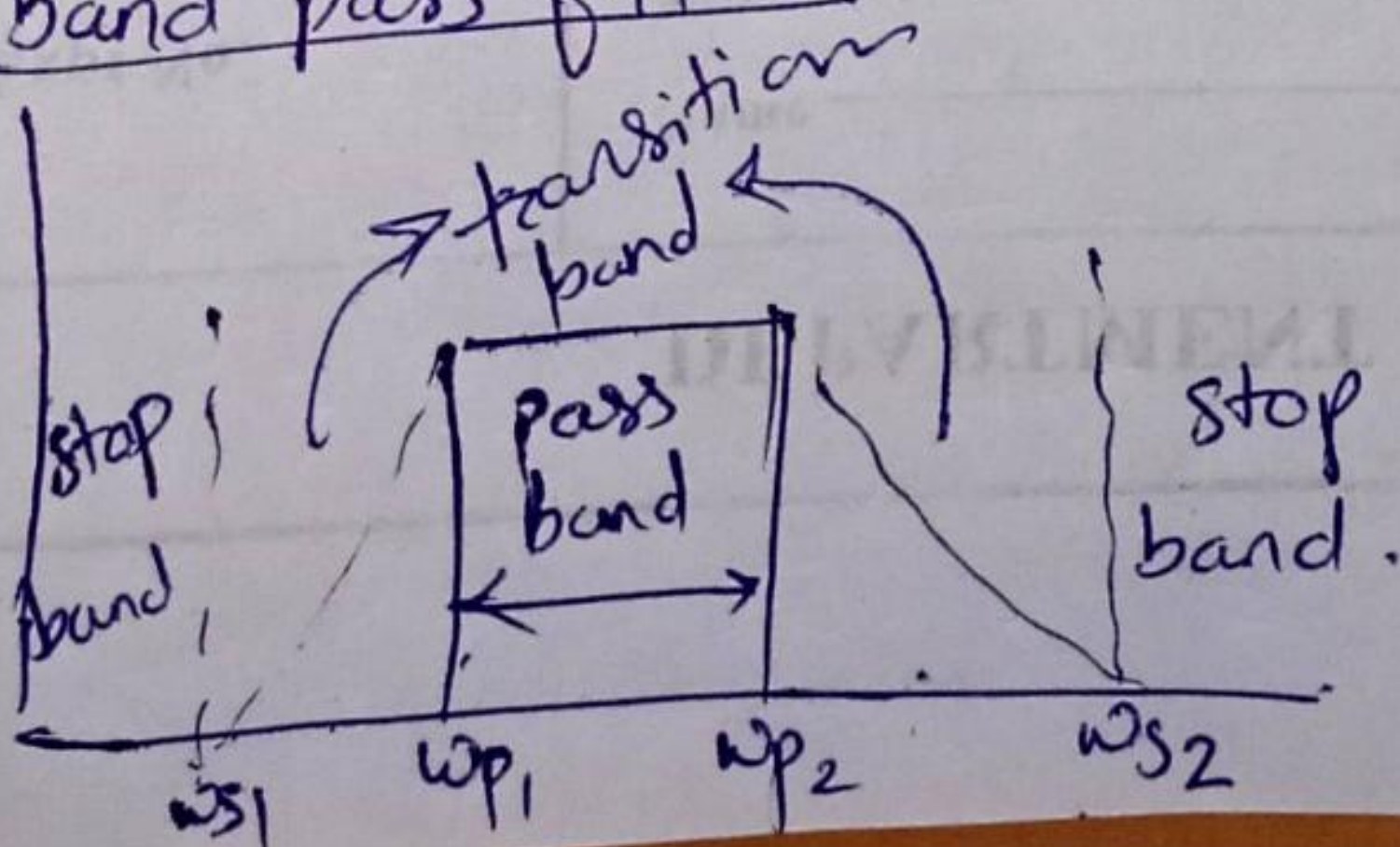
$$N \propto \frac{1}{(\omega_s - \omega_p)}$$

$\therefore$ as $(\omega_s - \omega_p) \downarrow$ $\rightarrow$ transition B.W. $\downarrow$ $\rightarrow$ ideal filter.

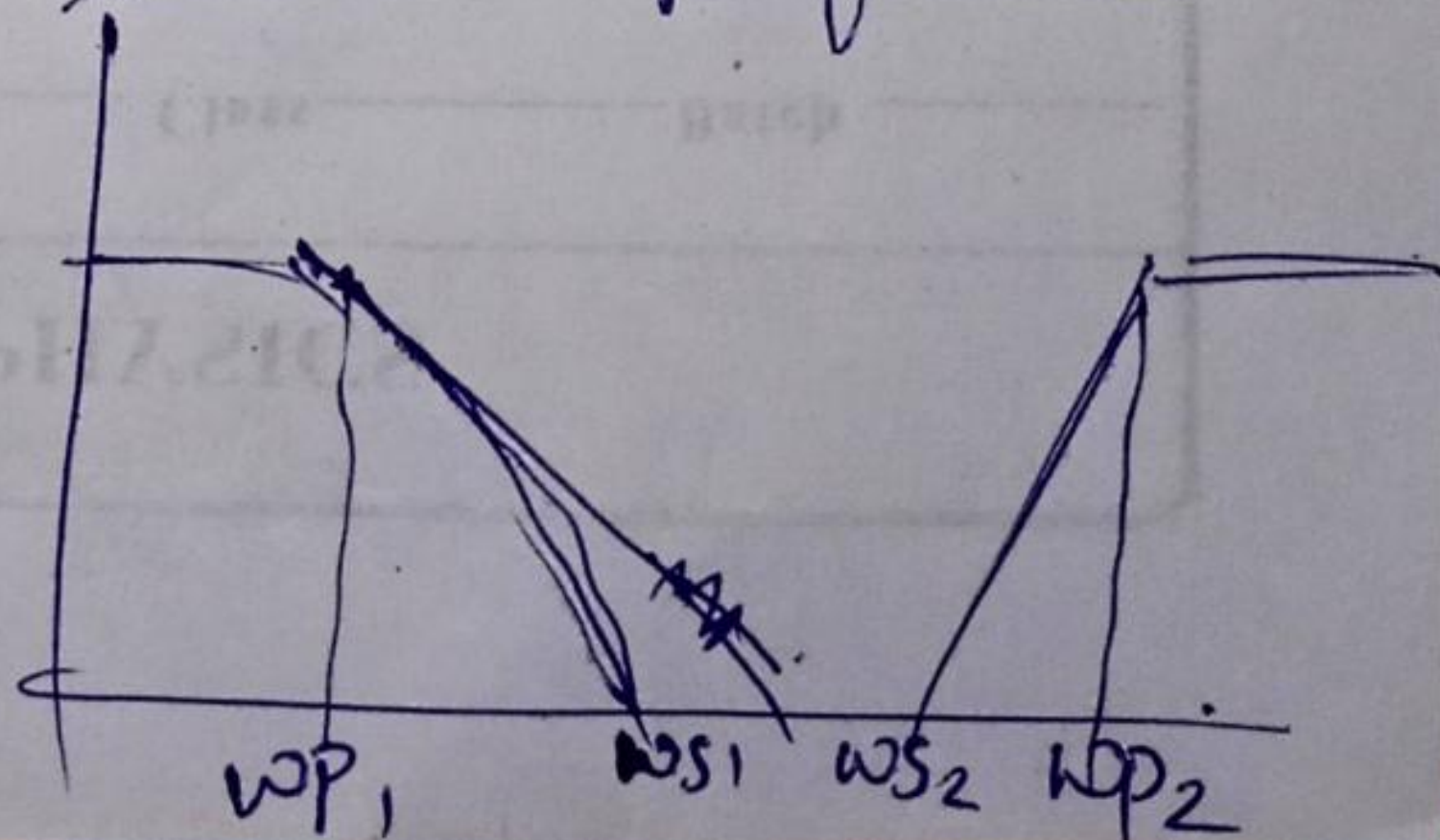
But at the same time $N \uparrow$ $\rightarrow$ computational complexity $\uparrow$, which we don't want.

To achieve ~~N~~ & N & $(\omega_s - \omega_p)$ small we need some different arrangement.

* band pass filter



* band stop filter



Filter order basically depends upon transition band with smallest width.

Q. Estimate the order of band pass filter.

F_{p1} = pass band edge frequencies = 0.35 KHz.

F_{p2} = ~~1~~ 1 KHz.

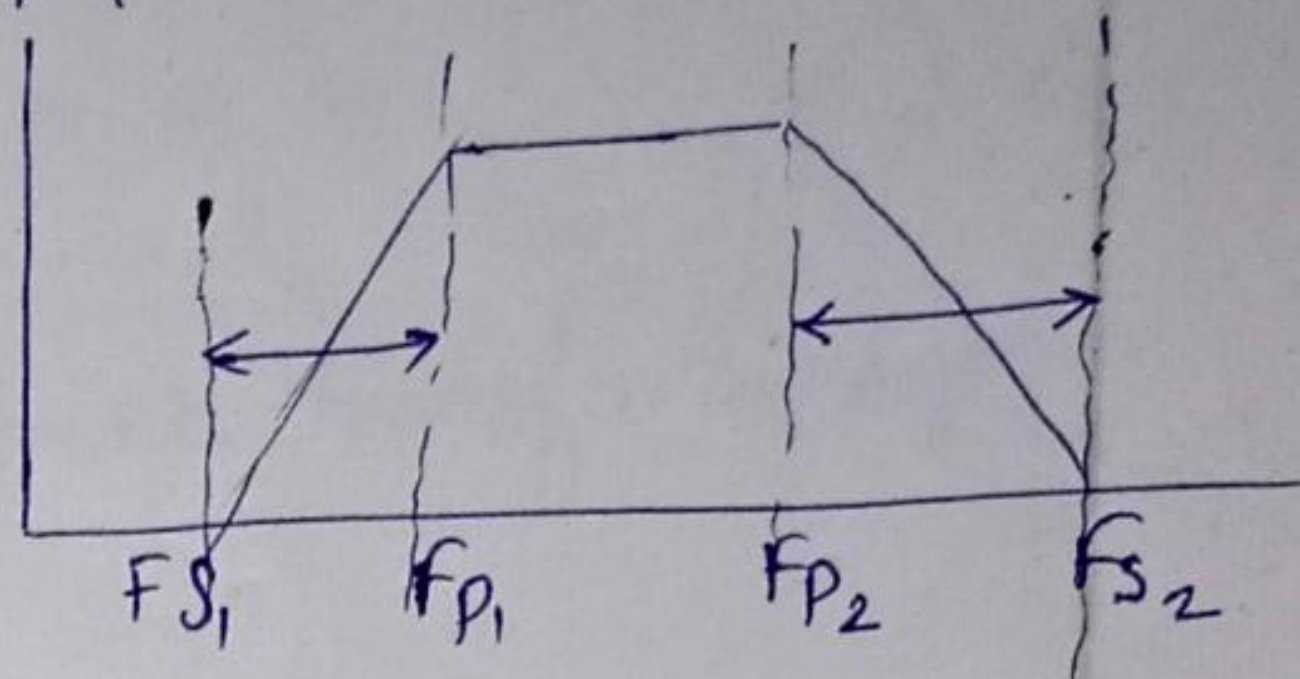
stop band edge frequencies = F_{s1} = 0.3 KHz.

F_{s2} = 1.1 KHz.

pass band ripple (δ_p) = 0.002

stopband ripple (δ_s) = 0.001

Sampling rate F_T = 10 KHz



transition Bandwidth 1 = $0.35 - 0.3$
 $= 0.05 \text{ KHz} = 50 \text{ Hz}$

transition Bandwidth 2 = $1.1 - 1 = 0.1 \text{ KHz} = 100 \text{ Hz}$

For further calculation choose the smallest B.W.

$$N = \frac{-20 \log(\sqrt{0.002(0.001)}) - 13}{14.6 \left(\frac{F_s - F_p}{F_T} \right)}$$

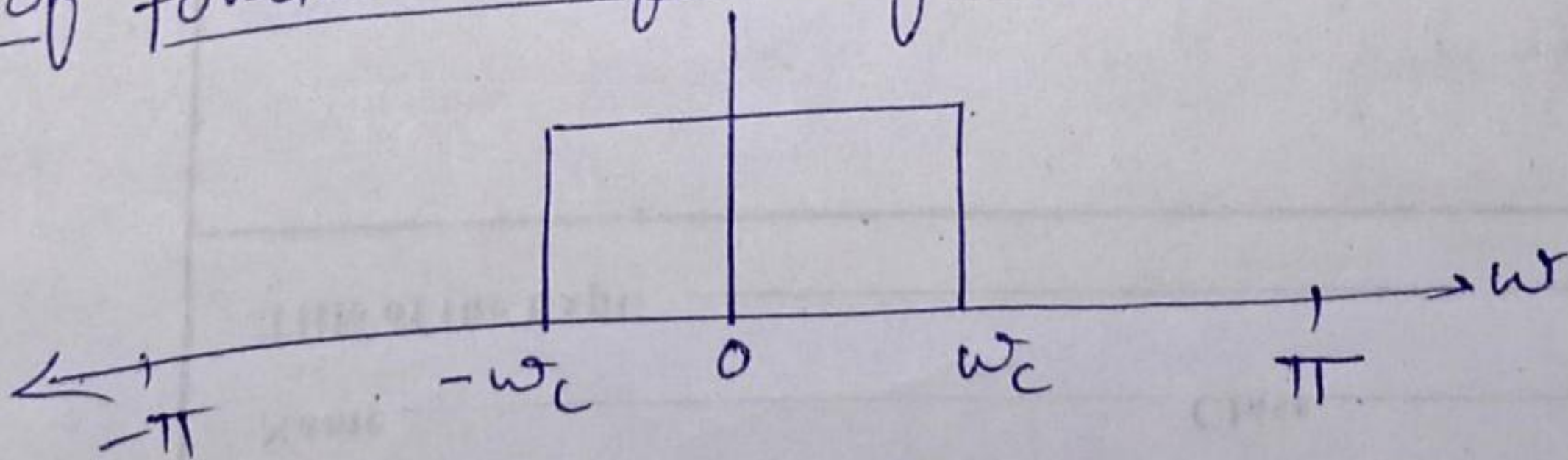
$$= \frac{43.9897}{14.6 \left(\frac{50}{10K} \right)} = 602.59 \approx 603$$

* Based on N (even or odd) the filters are classified as

1) Type 1 FIR filter $\rightarrow$ for even

2) Type 2 FIR filter $\rightarrow$ for odd.

* Plot of fourier transform of LPF.



$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & , 0 \leq |\omega| \leq \omega_c \\ 0 & , \omega_c \leq |\omega| \leq \pi \end{cases}$$

The inverse DTFT of $H_{LP}(e^{j\omega})$ is given as ~~$h_{LP}(n)$~~

$$\begin{aligned} h_{LP}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn} \right) \\ &= \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty \dots n \neq 0 \end{aligned}$$

If ~~$n=0$~~ For $n=0$, the inverse Fourier transform eqn reduces to

$$\begin{aligned} h_{LP}[0] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} d\omega = \frac{\omega_c}{\pi} \end{aligned}$$

$$h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & n = 0 \\ \frac{\sin \omega_c n}{\pi n}, & n \neq 0 \end{cases}$$

Methods to find ~~$h[n]$~~

- 1) Fourier series method
 - 2) windowed Fourier series
 - 3) Adjustable window functions.
- ~~Kaiser~~ Kaiser window is called as adjustable window fn.

$h(n)$ = Impulse response.

$h(n)$ = Inverse $H(e^{j\omega})$

High pass $\rightarrow h_{HP}[n]$

Band pass $\rightarrow h_{BP}[n]$

Band stop $\rightarrow h_{BS}[n]$

Impulse response

* Ideal LPF

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

$$h_{LP}(n) = \frac{\sin \omega_c n}{\pi n} \quad -\infty < n < \infty$$

absolutely. It is doubly infinite & not absolutely summable.
Hence it is non-realizable.

$$\hat{h}_{LP}[n] = \begin{cases} \frac{\sin(\omega_c(n-M))}{\pi(n-M)} & , 0 \leq n \leq N-1 \\ 0 & , \text{otherwise.} \end{cases}$$

* Ideal HPF. Impulse Response.

$$H_{HP}(e^{j\omega}) = \begin{cases} 0 & , |\omega| \leq \omega_c \\ 1 & , \omega_c \leq |\omega| \leq \pi \end{cases}$$

$$h_{HP}(n) = \begin{cases} 1 - \frac{\omega_c}{\pi} & \text{for } n=0 \\ -\frac{\sin(\omega_c n)}{\pi n} & \text{for } |n| > 0 \end{cases}$$

* Ideal Band Pass Filter.

$$H_{BP}(e^{j\omega}) = \begin{cases} 0 & , |\omega| < \omega_{c1} \\ 1 & , \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & , \omega_{c2} < |\omega| \leq \pi \end{cases}$$

$$h_{BP}(n) = \frac{\sin(\omega_{c2} n)}{\pi n} - \frac{\sin(\omega_{c1} n)}{\pi n} , |n| \geq 0$$

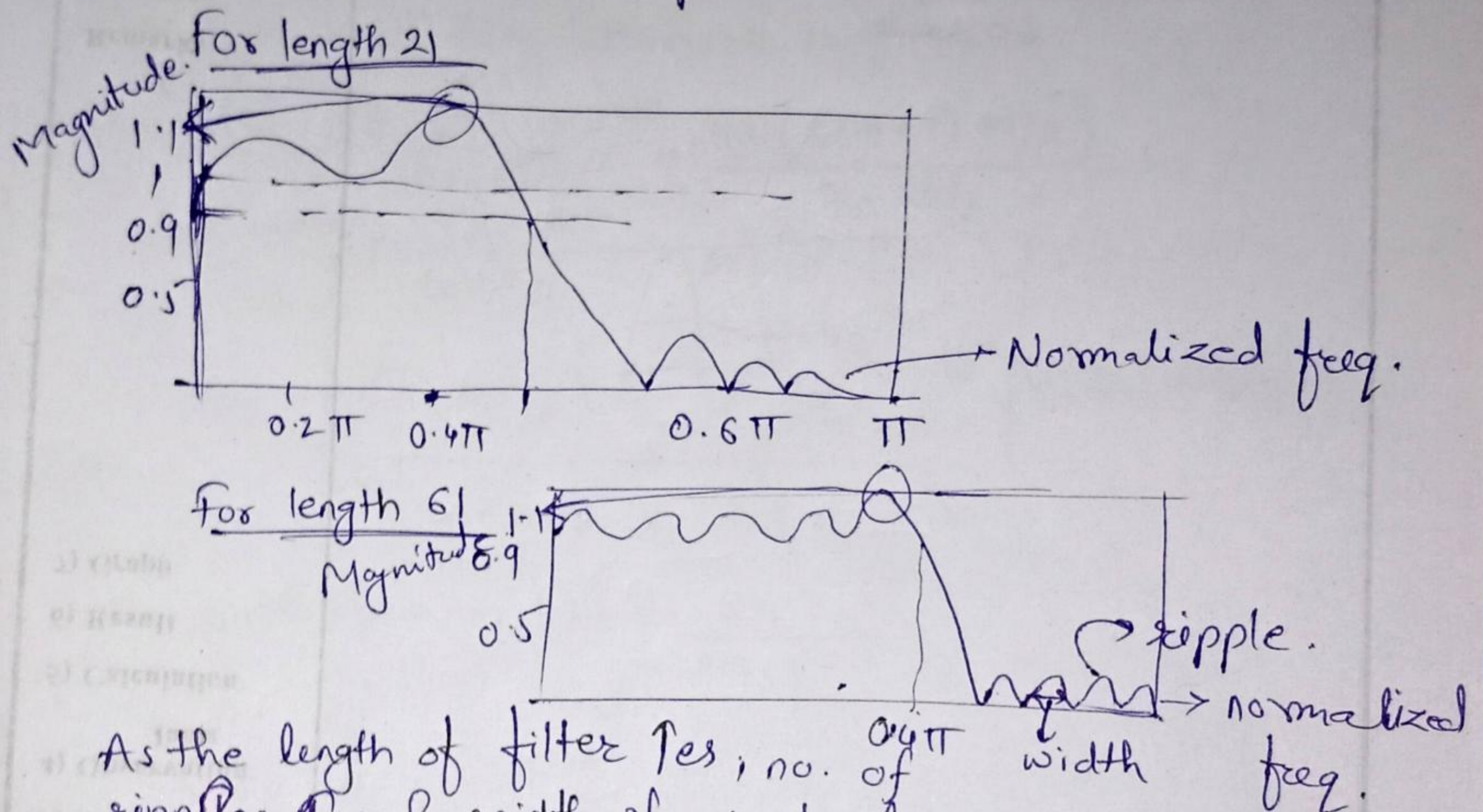
* Ideal Bandstop filter

$$H_{BS}(e^{j\omega}) = \begin{cases} 1 & , |\omega| \leq \omega_{c1} \\ 0 & , \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 1 & , \omega_{c2} \leq |\omega| \leq \pi \end{cases}$$

$$h_{BS}[n] = \begin{cases} 1 - \frac{(\omega_{c2} - \omega_{c1})}{\pi} & \text{for } n=0 \\ \frac{\sin(\omega_{c1} n) - \sin(\omega_{c2} n)}{\pi n} & \text{for } |n| > 0 \end{cases}$$

★ Gibb's Phenomenon

The causal ~~F~~ FIR filters obtained by simply truncating the impulse response coefficients of ideal filters exhibits oscillatory behaviour in their magnitude responses. This is known as Gibb's phenomenon.



As the length of filter N increases, no. of ripples N & width of ripple decreases.

Height of larger ripple remains the same, which occurs on both sides of cutoff freq.

∴ Magnitude of largest ripple is independent of filter length & it is approximately 11% of difference b/w pass band & stop band magnitudes of ideal filter

A truncation operation is obtained as multiplication by finite length window sequence $w[n]$ to $h_d[n]$

$$h_t[n] = h_d[n] \cdot w[n]$$

Simplest window is Rectangular window.

$$w_R[n] = \begin{cases} 1 & , -M \leq n \leq M \\ 0 & , \text{otherwise} \end{cases}$$

$$\text{Length of } h_t[n] = 2M + 1$$

$h_d[n] \rightarrow$ Fourier series method.

$w_R[n] \rightarrow$ window.

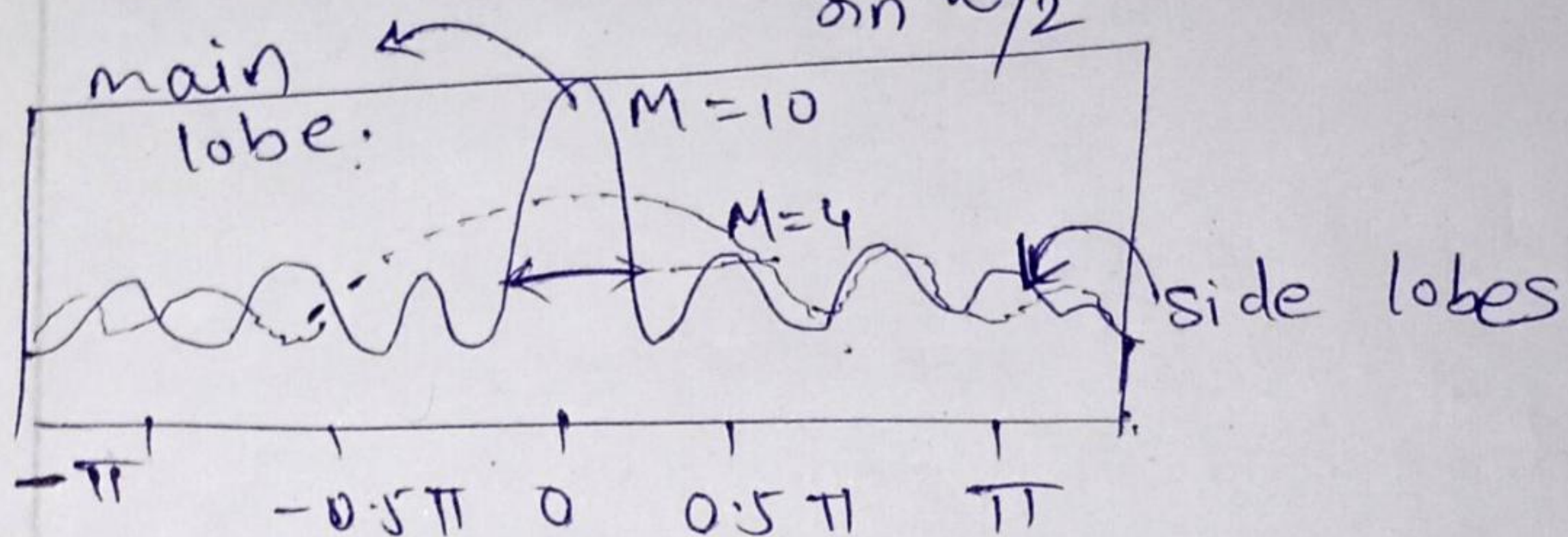
$$h_t[n] = h_d[n] \cdot w[n]$$

The presence of oscillatory behaviour in Fourier transform of truncated Fourier series of an ideal filter is due to 2 reasons:

- 1) The impulse response of ideal filter is infinitely long & not absolutely summable. Hence the filter is unstable.
- 2) The Rectangular window has an abrupt transition to zero.

So the oscillatory behaviour is given as.

$$\Psi_R(e^{j\omega}) = \sum_{n=-M}^M e^{-j\omega n} = \frac{\sin[(2M+1)\omega/2]}{\sin \omega/2}$$



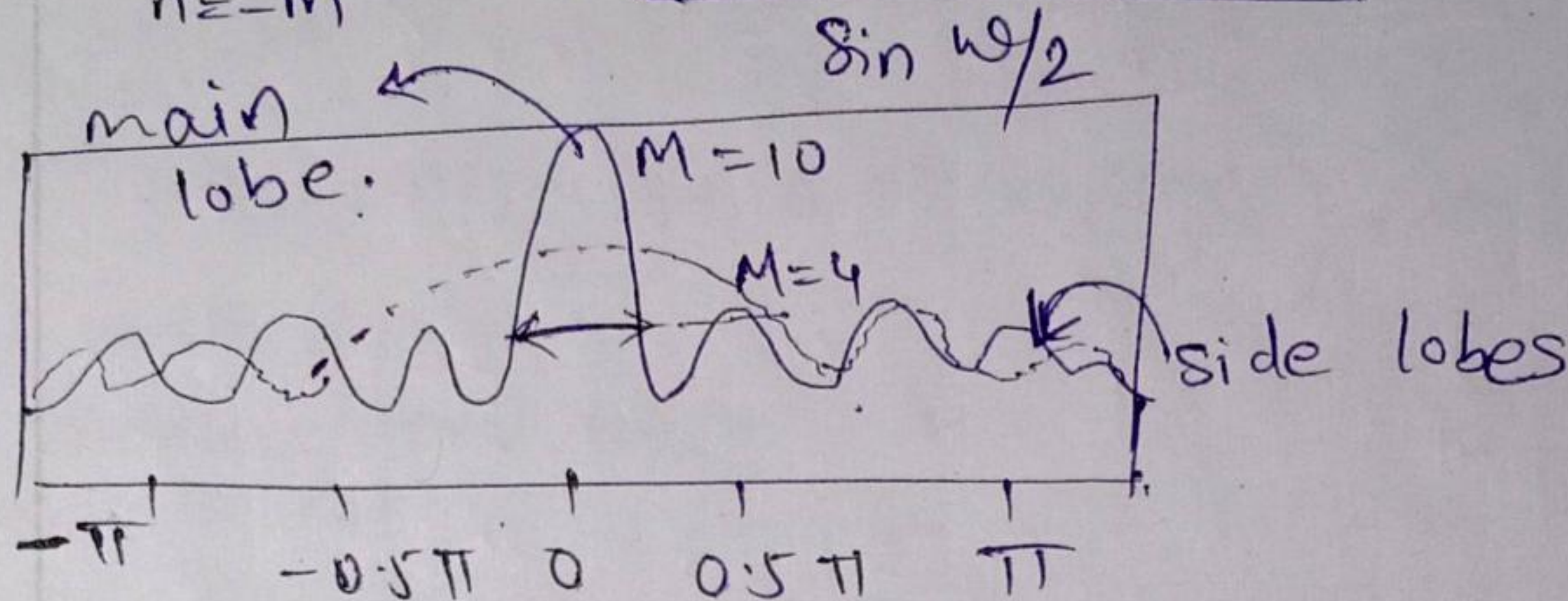
width of main lobe = $\frac{4\pi}{2M+1}$

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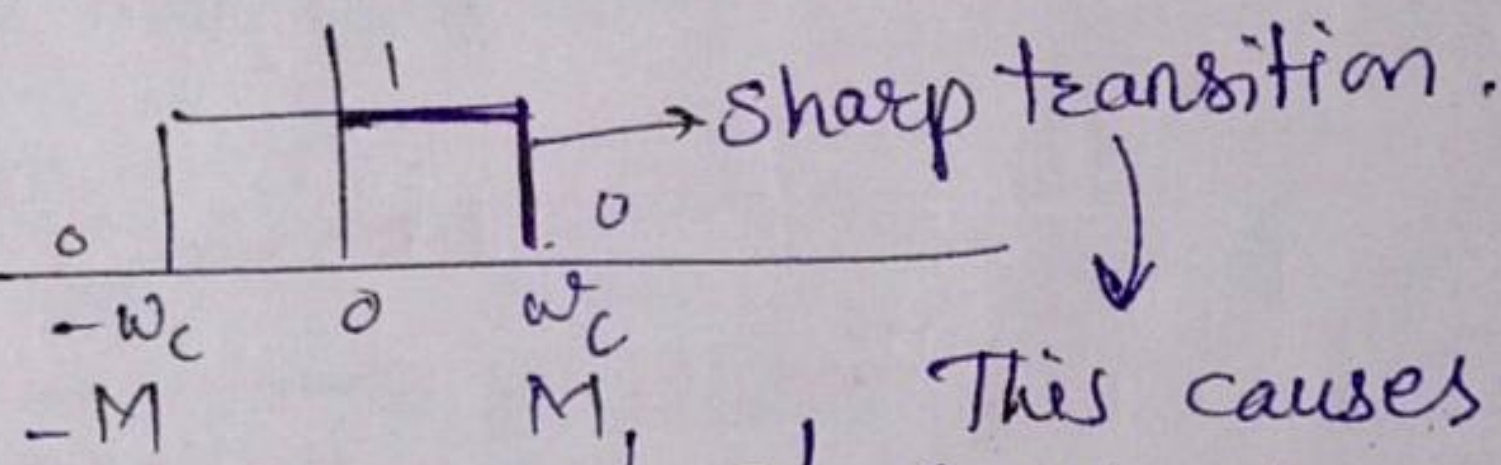
$$\Psi_R(e^{j\omega}) = \sum_{n=-M}^M e^{-j\omega n} = \frac{\sin[(2M+1)\omega/2]}{\sin \omega/2}$$



$$\text{width of main lobe} = \frac{4\pi}{2M+1}$$

Fixed window function.

Rectangular window
It has abrupt transition outside range $(-M \text{ to } M)$.
Here we obtained ripples in both pass band & stop band.



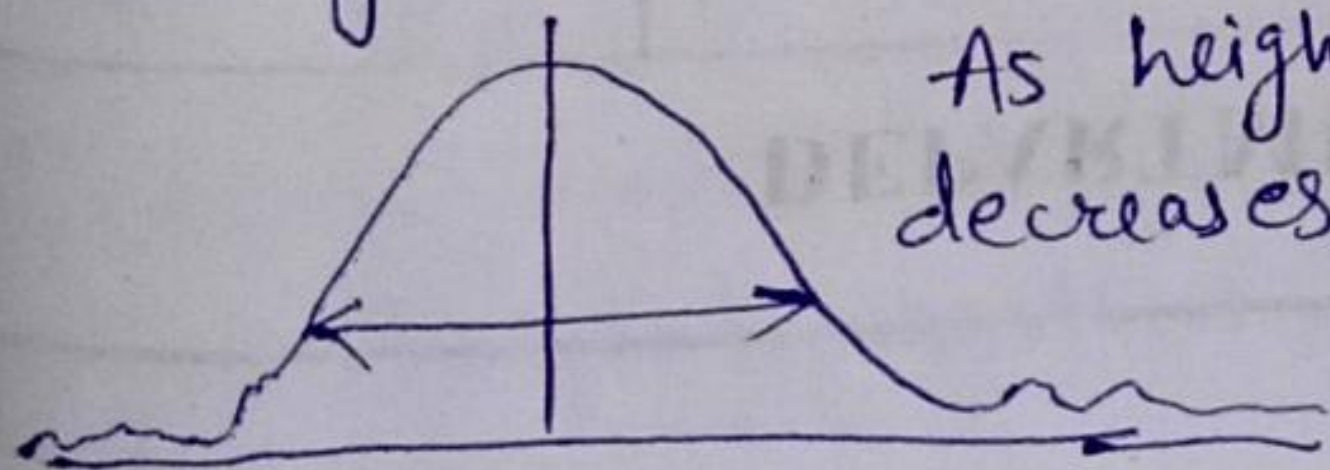
This causes appearance of Gibb's phenomenon.

Gibb's phenomenon can be reduced by using a window that tapers smoothly to zero at each end.

② Providing smooth transition from pass band to stop band

Use of tapered window causes the height of side lobe to diminish with a corresponding increase in main lobe width resulting in wider transition at discontinuity.

As height of main lobe increases, side lobes decreases.



* Bartlett window function.

$$w[n] = 1 - \frac{|n|}{M+1}, \quad -M \leq n \leq M$$

* Hann window (Raised cosine window)

$$w[n] = \frac{1}{2} \left[1 + \cos\left(\frac{\pi n}{M}\right) \right], \quad -M \leq n \leq M$$

* Hamming window

$$w[n] = 0.54 + 0.46 \cos\left(\frac{\pi n}{M}\right); \quad -M \leq n \leq M$$

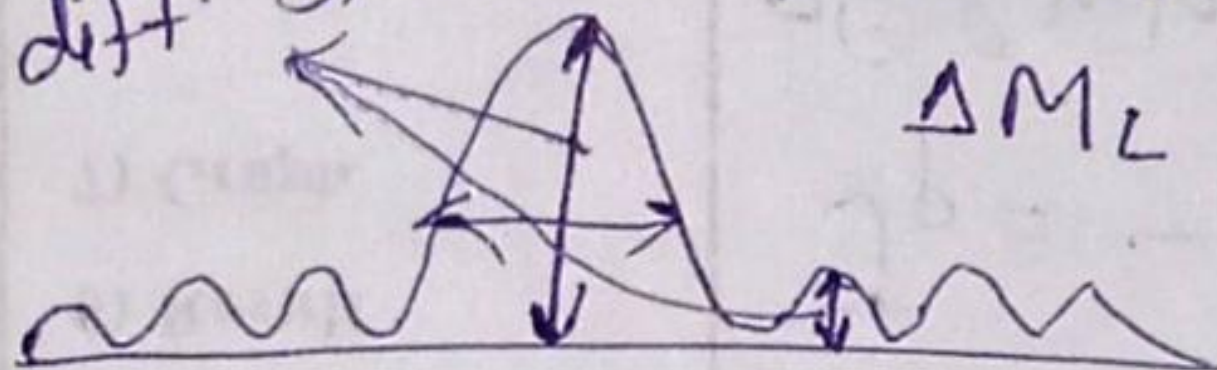
* Blackmann window

$$w[n] = 0.42 + 0.5 \cos\left(\frac{\pi n}{M}\right) + 0.08 \cos\left(\frac{2\pi n}{M}\right), \quad -M \leq n \leq M$$

→ Main lobe width (ΔM_L)

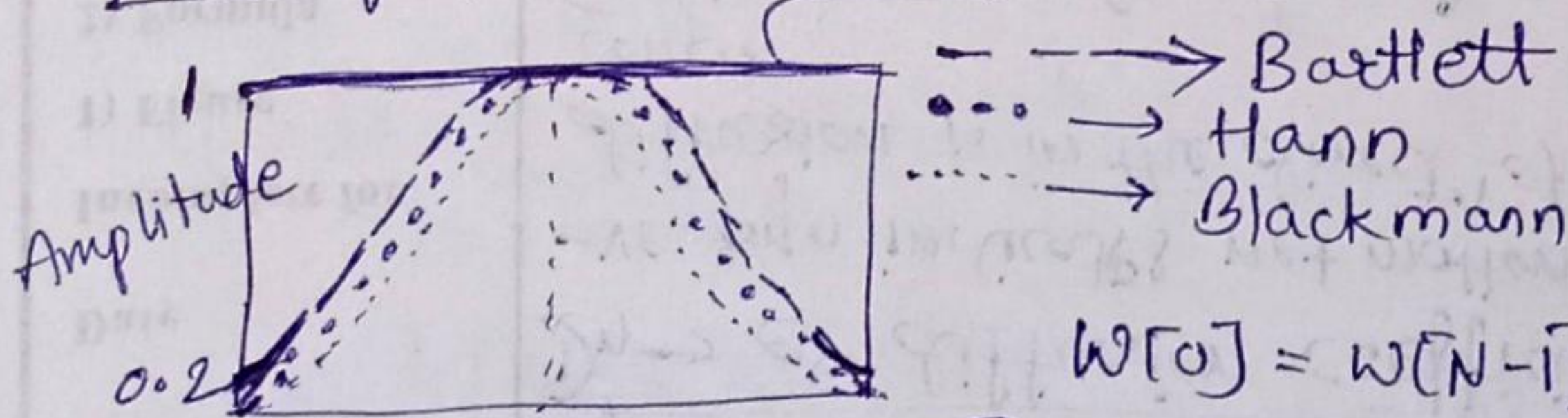
→ Relative sidelobe level (A_{sl})

diff. b/w these 2 is A_{sl} .



Relative sidelobe level is the difference in dB b/w the amplitudes of largest sidelobe & the main lobe

Plots of fixed windows. Rectangular:



$$w[0] = w[N-1] = 0$$

These 3 windows are of length $(N-2)$

* To ensure fast transition from pass band to stop band window should have very small main lobe width.

* On the other hand to reduce the pass band & stop band ripples 'S', the area under the side lobes should be very small.

- if Area under side band $\uparrow$ ripples in side band $\uparrow$.

- For sharp transition, less ripples must be there in side band.

The transition bandwidth is given by,

$$\Delta\omega = \frac{C}{M} \quad C = \text{constant}.$$

* Fourier Series Method ~~for~~ ^{of} designing FIR filters.

The frequency response $H(e^{j\omega})$ is periodic in 2π . From Fourier series analysis we know that any periodic fn. can be explained as linear combination of complex exponentials. Therefore the desired frequency response of FIR filter is represented by Fourier series.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n} \quad \text{--- (1)}$$

$h_d(n)$ are Fourier coefficients $\rightarrow$ desired impulse response sequence of filter.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \quad \text{--- (2)}$$

Z transform of sequence is given as,

$$H(z) = \sum_{n=-\infty}^{\infty} h_d(n) z^{-n} \quad \text{--- (3)}$$

~~$h(n)$~~ Truncated as $h(n) = \begin{cases} h_d(n), & \text{for } |n| \leq (N-1)/2 \\ 0, & \text{otherwise} \end{cases}$

$$H(z) = \sum_{n=-(N-1)/2}^{(N-1)/2} h(n) z^{-n}$$

$$= h\left[\frac{N-1}{2}\right] z^{-\frac{(N-1)}{2}} + \dots + (h_1) z^{-1} + h(0) + \dots$$

$$h(-1) z^1 + h(-2) z^2 + \dots + h\left[-\frac{(N-1)}{2}\right] z^{\frac{(N-1)}{2}}$$

$$\therefore H(z) = h[0] + \sum_{n=1}^{(N-1)/2} [h(n) z^{-n} + h(-n) z^n]$$

but, for symmetrical impulse response.

$$H(z) = h[0] + \sum_{n=1}^{(N-1)/2} h[n] [z^n + z^{-n}]$$

But this fn. is not realizable. $\therefore$ Multiply it by $z^{-(N-1)/2}$

∴ Realization transfer fn. $\Rightarrow$

$$H'(z) = z^{-(N-1)/2} \cdot H(z)$$

① Symmetrical impulse response 'N' odd.

High Pass, low Pass, Band pass, Band stop.

Magnitude Response.
 $|H(e^{j\omega})|$

$$\sum_{n=0}^{(N-1)/2} a(n) \cos \omega n$$

where

$$a_0 = h\left[\frac{N-1}{2}\right]$$

$$a(n) = 2h\left[\frac{N-1}{2} - n\right]$$

Phase Response

$$-\alpha\omega + \theta$$

$$\theta = 0 \text{ for } H(e^{j\omega}) > 0$$

$$\theta = \pi \text{ for } H(e^{j\omega}) < 0$$

② Symmetrical impulse response 'N' even

low pass,

High pass.

$$\sum_{n=0}^{N/2} b(n) \cos\left[n - \frac{1}{2}\right]\omega$$

$$-\alpha\omega + \theta$$

$$\theta = 0 \text{ for } H(e^{j\omega}) > 0$$

$$\theta = \pi \text{ for } H(e^{j\omega}) < 0$$

Q] Design an ideal low pass filter with frequency response $H_d(e^{j\omega}) = 1$ for $-\pi/2 \leq \omega \leq \pi/2$ &

$= 0$ for $\pi/2 \leq \omega \leq \pi$. Find values of

$h(n)$ for $N=11$. Plot Amplitude response.

(if value of N is not given then calculate it with Kaiser's formula).

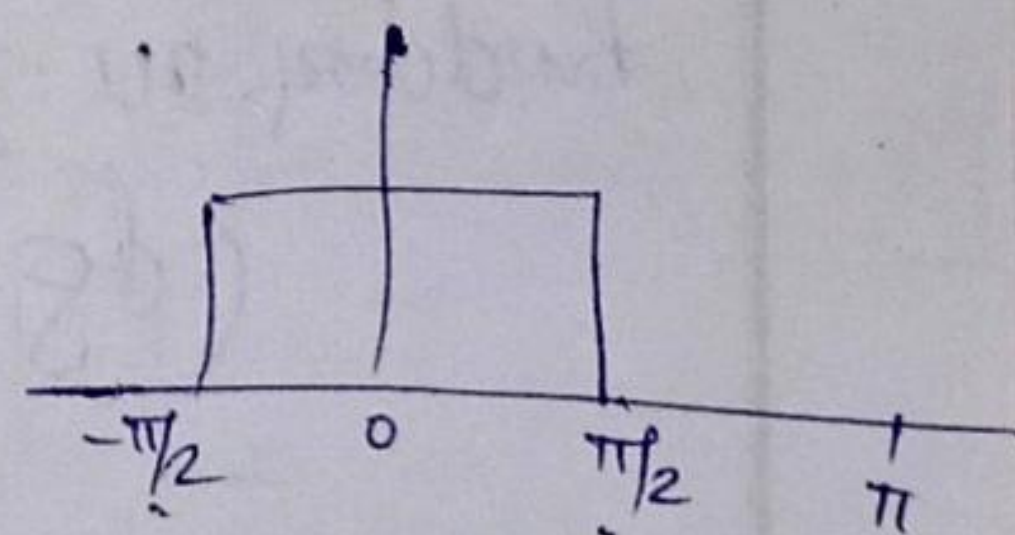
$$\rightarrow h_d(n) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} (1) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi jn} \left[e^{j\pi/2 n} - e^{-j\pi/2 n} \right]$$

$$= \frac{1}{2\pi jn} \left[\cos \pi/2 n + j \sin \pi/2 n - \cos \pi/2 n + j \sin \pi/2 n \right]$$

$$= \frac{j 2 \sin \pi/2 n}{2\pi jn} = \frac{\sin(\pi/2 n)}{\pi n}$$



truncating $h_d(n)$ to 11 samples.

$$h(n) = \frac{\sin \pi/2 n}{\pi n}, \text{ for } |n| \leq 5 \Rightarrow h(1) = h(-1)$$

$$= 0, \text{ otherwise.}$$

$$h[0] = \lim_{n \rightarrow 0} \frac{\sin \pi/2 n}{\pi n} = \frac{1}{2}$$

$$\text{put } n=1 \quad h(1) = h(-1) = \frac{\sin \pi/2}{\pi} = \frac{1}{\pi} = 0.3183$$

$$n=2 \quad h(2) = h(-2) = \frac{\sin \pi}{2\pi} = 0$$

$$n=3 \quad h(3) = h(-3) = \frac{\sin 3\pi/2}{3\pi} = \frac{-1}{3\pi} = -0.106$$

$$n=4 \quad h(4) = h(-4) = \frac{\sin 2\pi}{4\pi} = 0$$

$$n=5 \quad h(5) = h(-5) = \frac{\sin 5\pi/2}{5\pi} = \frac{1}{5\pi} = 0.06366$$

$$H[z] = h(0) + \sum_{n=1}^{(N-1)/2} h[n] (z^n + z^{-n})$$

$$= \frac{1}{2} + [0.3183(z^1 + z^{-1}) - 0.106(z^3 + z^{-3})$$

$$+ 0.06366(z^5 + z^{-5})]$$

The transfer fn. of realizable filter is,

$$\bar{H}(z) = z^{-5} [0.5 + 0.3183(z^1 + z^{-1}) - 0.106(z^3 + z^{-3}) + 0.06366(z^5 + z^{-5})]$$

$$\bar{H}(z) = 0.5 z^{-5} + 0.3183 [z^{-4} + z^{-6}] - 0.106 [z^{-2} + z^{-8}] + 0.06366 [z^0 + z^{-10}]$$

$$\bar{H}(z) = 0.06366 - 0.106 z^{-2} + 0.3183 z^{-4} + 0.5 z^{-5} + 0.3183 z^{-6} - 0.106 z^{-8} + 0.06366 z^{-10}$$

The filter coefficients are.

$$\left. \begin{array}{l} h(0) = h(10) = 0.06366 \\ h(1) = h(9) = 0 \end{array} \right\} \begin{array}{l} h(2) = h(8) = -0.106 \\ h(3) = h(7) = 0 \end{array}$$

Example of Kaiser window is adjustable window.

$$W = V^{\alpha} E - (E^{\alpha} - E^1) / KL$$

$$W = V^{\alpha} E - (E^{\alpha} - E^1) / KL$$

for $E^{\alpha} > E^1$: $\delta = V^{\alpha} E - (E^{\alpha} - E^1) / KL$

$$[1 - t(E^1)] = \frac{E - (E^{\alpha} - E^1) / KL}{1 + (E^{\alpha} - E^1) / KL}$$

$$1 - t(E^1) = 1 - \frac{E - (E^{\alpha} - E^1) / KL}{1 + (E^{\alpha} - E^1) / KL}$$

$$W = V^{\alpha} [1 - t(E^1)]$$

for $E^{\alpha} < E^1$: $\delta = V^{\alpha} E - (E^{\alpha} - E^1) / KL$

$$W = V^{\alpha} [1 - t(E^1)]$$

for $E^{\alpha} < E^1$: $\delta = V^{\alpha} E - (E^{\alpha} - E^1) / KL$

$$W = V^{\alpha} E - (E^{\alpha} - E^1) / KL$$

$$KL = 0.0096 \alpha$$

$$t(E^{\alpha}) = \frac{(E^{\alpha} - E^1) / KL}{1 + (E^{\alpha} - E^1) / KL}$$

ex: for $\alpha = 12$: $KL = 0.0096 \times 12 = 0.1152$

$$W = V^{\alpha} t(E^{\alpha})$$

Comparison with other windows

Time of the exam

Name

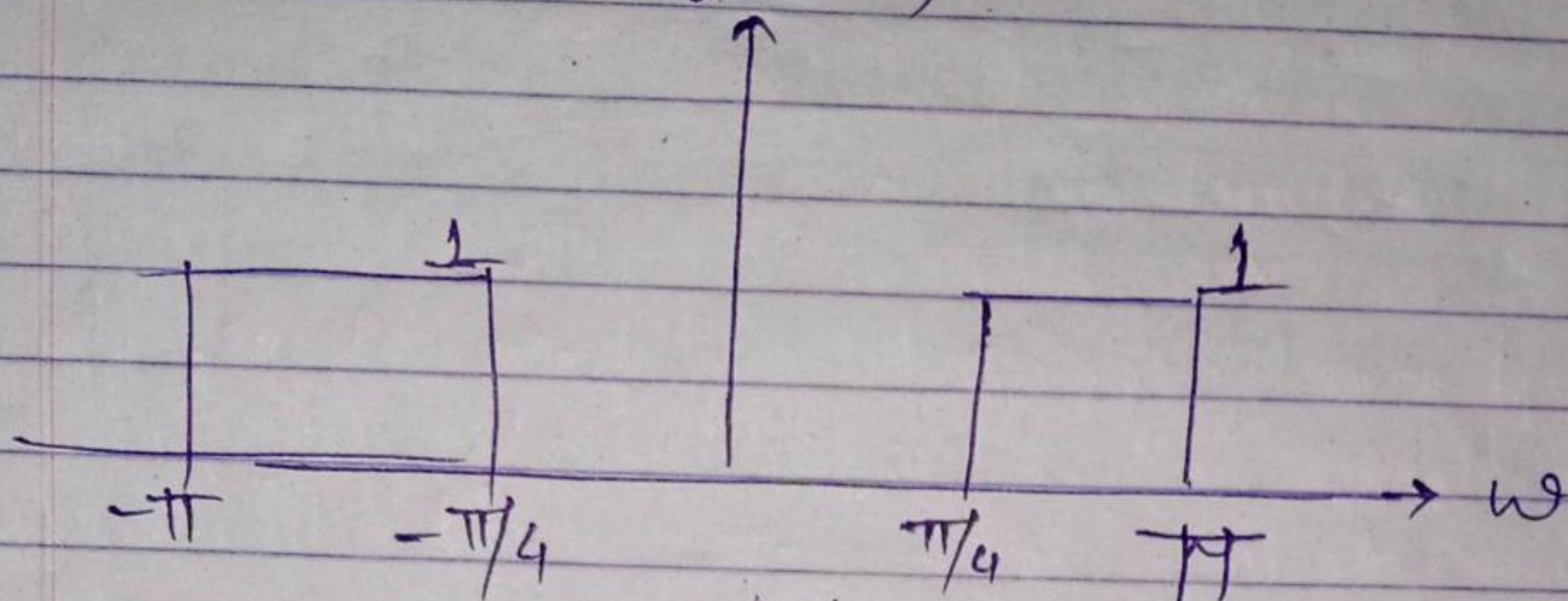
Class

Date

DEPARTMENT OF PHYSICS

11/20

Q.] Design an ideal HPF with frequency response
 $H_d(e^{j\omega}) = 1$ for $\pi/4 \leq |\omega| \leq \pi$
 $= 0$ for $|\omega| < \pi/4$
 find $h(n)$ for $N=11$



$$H_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega \right]$$

$$H_d(n) = \frac{1}{2\pi j n} \left[e^{-j\pi n/4} - e^{-j\pi n} + e^{j\pi n} - e^{j\pi n/4} \right]$$

$$H_d(n) = \frac{1}{\pi n} \left[\sin(n\pi) - \sin(n\pi/4) \right]$$

for $n=0$

$$h(0) = \lim_{n \rightarrow 0} \frac{\sin \pi n}{\pi n} - \lim_{n \rightarrow 0} \frac{\sin(\pi/4)n}{\pi n} = 1 - \frac{1}{4}$$

$$h(0) = 0.75$$

$$h(1) = h(-1) = \frac{\sin \pi - \sin \pi/4}{\pi}$$

$$= -0.225$$

$$h(2) = h(-2) = \frac{\sin 2\pi - \sin \pi/2}{2\pi} = -0.159$$

$$h(3) = h(-3) = \frac{\sin 3\pi - \sin 3\pi/4}{3\pi} = -0.075$$

$$h(4) = h(-4) = \frac{\sin 4\pi - \sin \pi}{4\pi} = 0$$

$$h(5) = h(-5) = \frac{\sin 5\pi - \sin 5\pi/4}{5\pi} = 0.045$$

Transfer fn.

$$H(z) = h(0) + \sum_{n=1}^{(N-1)/2} [h(n)(z^n + z^{-n})]$$

$$= 0.75 + \sum_{n=1}^5 [h(n)(z^n + z^{-n})]$$

$$H(z) = 0.75 - 0.225(z^1 + z^{-1}) - 0.159(z^2 + z^{-2}) - 0.075(z^3 + z^{-3}) + 0.045(z^5 + z^{-5})$$

The transfer fn. of realizable filter is

$$\tilde{H}(z) = z^{-5}(H(z))$$

$$= z^{-5} (0.75 - 0.225(z^1 + z^{-1}) - 0.159(z^2 + z^{-2}) - 0.075(z^3 + z^{-3}) + 0.045(z^5 + z^{-5}))$$

$$\tilde{H}(z) = 0.045 - 0.075z^{-2} - 0.159z^{-3} - 0.225z^{-4} + 0.75z^{-5} - 0.225z^{-6} - 0.159z^{-7} - 0.075z^{-8} + 0.045z^{-10}$$

$\therefore$ coefficients of causal filter are

$$h(0) = h(10) = 0.045$$

$$h(1) = h(9) = 0$$

$$h(2) = h(8) = 0.075$$

$$h(3) = h(7) = -0.159$$

$$h(4) = h(6) = -0.225$$

$$h(5) = 0.75$$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{(N-1)/2} a_n \cos \omega n \quad \text{where}$$

$$a_0 = h\left[\frac{N-1}{2}\right] \quad a(n) = 2h\left[\frac{N-1}{2} - n\right]$$

$$a(1) = 2h[5-1] = 2h[4] = -0.45$$

$$a(2) = -0.318$$

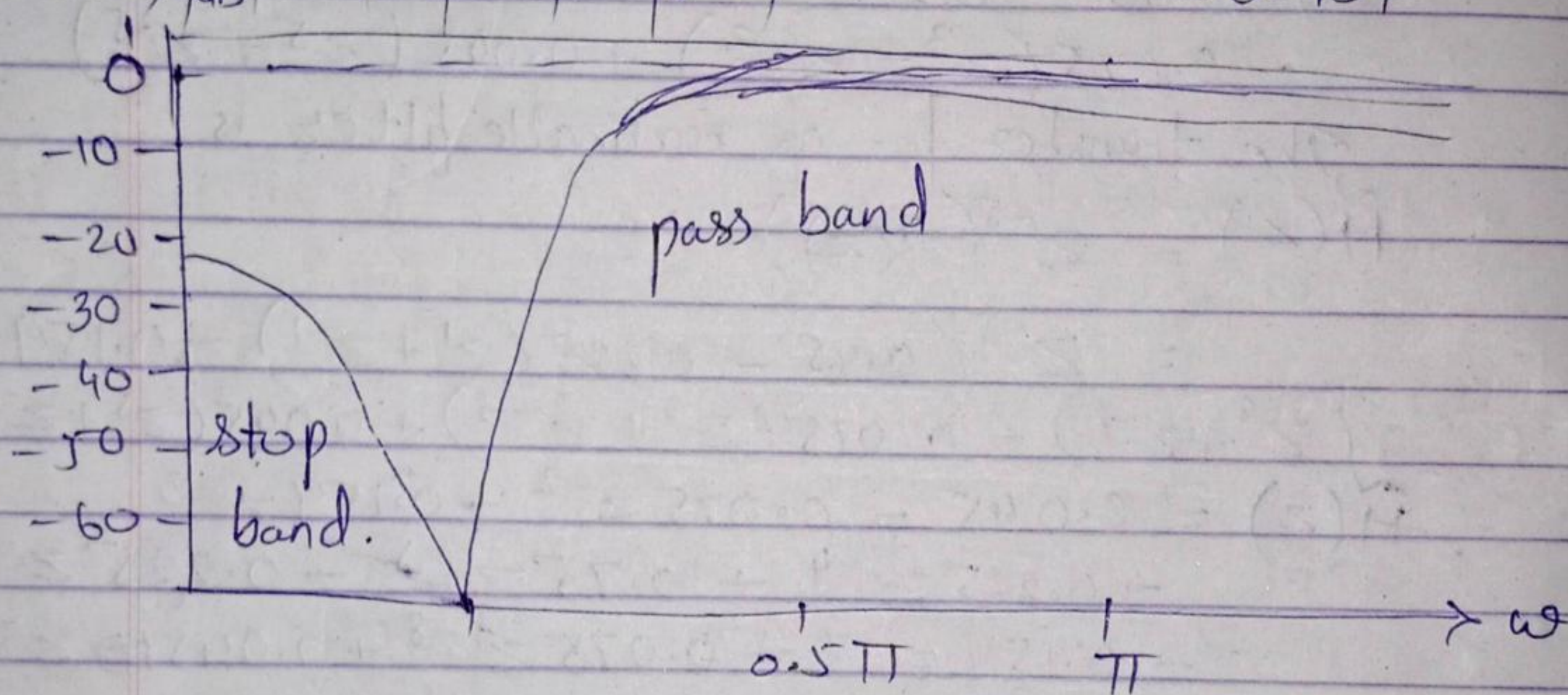
$$a(3) = -0.15$$

$$a(4) = 0$$

$$a(5) = 0.09$$

$$\begin{aligned}\bar{H}(e^{j\omega}) &= a(0) + a(1)\cos\omega + a(2)\cos 2\omega \\ &\quad + a(3)\cos 3\omega + a(4)\cos 4\omega + a(5)\cos 5\omega \\ \tilde{H}(e^{j\omega}) &= 0.75 - 0.45\cos\omega - 0.318\cos 2\omega \\ &\quad - 0.15\cos 3\omega + 0.09\cos 5\omega \\ |\bar{H}(e^{j\omega})| &= 20 \log |\tilde{H}(e^{j\omega})|\end{aligned}$$

ω (in degrees)	0	10	20	30	80	...	180
$\tilde{H}(e^{j\omega})$	-0.08	-0.066	-0.0088	0.122	1.1		0.94
$ \bar{H}(e^{j\omega}) _{dB}$	-22	-23.62	-41.7	-18.2	0.95		-0.937



Q. Design the same filter with same specification using ~~Hanning~~ Hanning window.

→
$$W_H(n) = 0.5 + 0.5 \cos \frac{2\pi n}{N-1} \quad \text{for } -\left(\frac{N-1}{2}\right) \leq n \leq \left(\frac{N-1}{2}\right)$$

∴ coefficients using fourier series method. $\left(\frac{N-1}{2}\right)$

$$\begin{aligned}W_H[0] &= 0.5 + 0.5 = 1 \\ W_H[1] &= W_H[-1] = 0.5 + 0.5 \cos \pi/5 = 0.9045 \\ W_H[2] &= W_H[-2] = 0.5 + 0.5 \cos 2\pi/5 = 0.655 \\ W_H[3] &= W_H[-3] = 0.5 + 0.5 \cos 3\pi/5 = 0.345 \\ W_H[4] &= W_H[-4] = 0.5 + 0.5 \cos 4\pi/5 = 0.0945 \\ W_H[5] &= W_H[-5] = 0\end{aligned}$$

The filter coefficients using ~~no~~ hamming window are

$$h(n) = h_d(n) w_H(n)$$

$$h(0) = h_d(0) w_H(0) = 0.75$$

$$h(-1) = h(1) = h_d(1) w_H(1) = -0.204$$

$$h(2) = h(-2) = h_d(2) w_H(2) = -0.104$$

$$h(3) = -0.026$$

$$h(4) = 0$$

$$h(5) = 0$$

Transfer fn. of filter

$$H[z] = h(0) + \sum_{n=1}^5 h(n) [z^n + z^{-n}]$$

$$\tilde{H}[z] = 0.75 - 0.204(z^{-1} + z^1) - 0.104(z^2 + z^{-2}) - 0.026(z^3 + z^{-3})$$

multiply with $z^{-\frac{(N-1)}{2}}$

$$\therefore \tilde{H}[z] = -0.026 z^{-2} - 0.104 z^{-3} - 0.204 z^{-4} + 0.75 z^{-5} - 0.204 z^{-6} - 0.104 z^{-7} - 0.026 z^{-8}$$

$$h(0) = h(1) = h(9) = h(10) = 0$$

$$h(2) = h(8) = -0.026$$

$$h(3) = h(7) = -0.104$$

$$h(4) = h(6) = -0.204$$

$$h(5) = 0.75$$

Use these coefficients to calculate $a(n)$

$$\tilde{H}(e^{j\omega}) = 0.75 - 0.408 \cos \omega - 0.208 \cos 2\omega - 0.052 \cos 3\omega$$

$$H[z] = h(0) + \sum_{n=1}^5 [h(n) (z^{-n} + z^n)]$$

$$H[z] = 0.75 - 0.2052(z^{-1} + z^1) - 0.1084(z^2 + z^{-2}) - 0.03(z^{-3} + z^3) + 0.0036(z^{-5} + z^5)$$

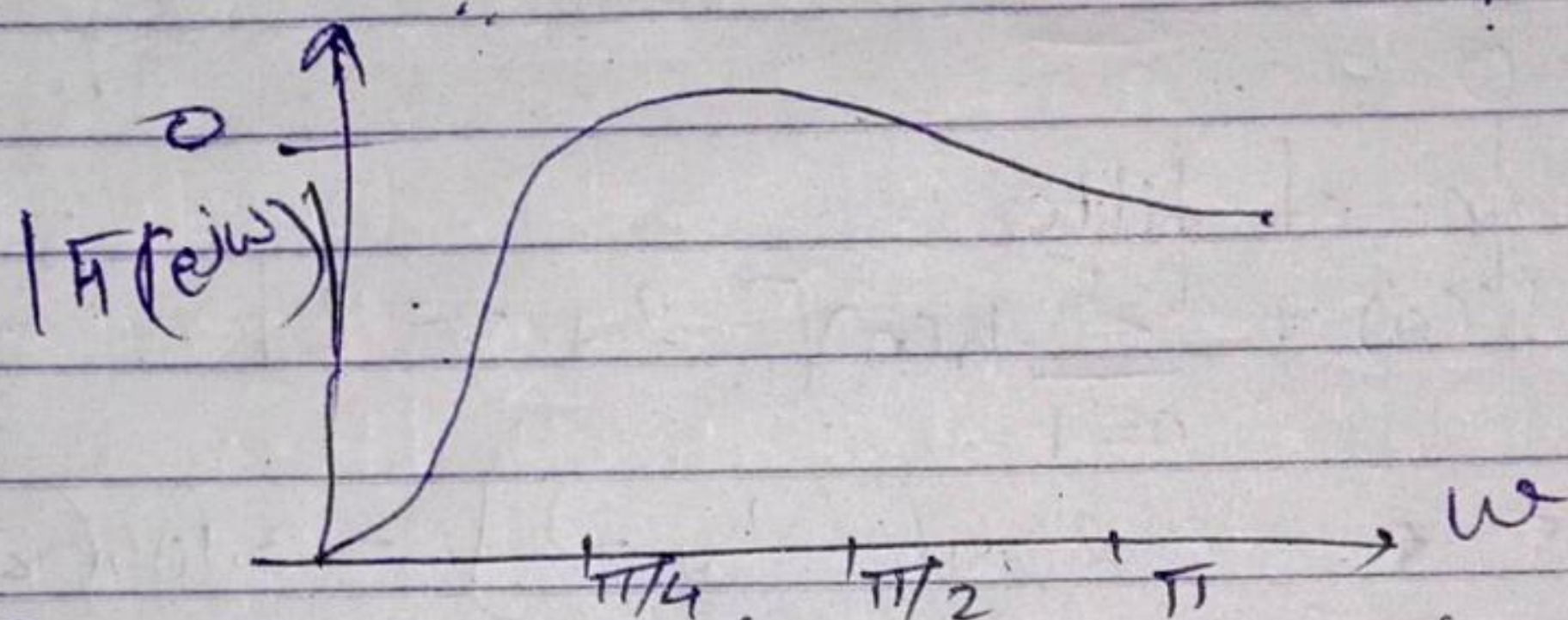
Multiply by z^{-5} to make it realizable

$$\therefore \tilde{H}[z] = 0.75 z^{-5} - 0.2052 z^{-6} - 0.2052 z^{-4} - 0.1084 z^{-7} - 0.1084 z^{-3} - 0.03 z^{-8} - 0.03 z^{-2} + 0.0036 z^{-10} + 0.0036$$

$$H(e^{j\omega}) = 0.75 - 0.408 \cos \omega - 0.208 \cos 2\omega - 0.052 \cos 3\omega$$

$$|H(e^{j\omega})|_{dB} = 20 \log_{10} |H(e^{j\omega})|$$

ω	$H(e^{j\omega})$	$ H(e^{j\omega}) _{dB}$
0	0.082	-21.72
15	0.139	-17.14
30	0.292	-10.67
$\vdots$	$\vdots$	$\vdots$



★ ~~fixed window~~ Kaiser window is given by

$$w_k[n] = \frac{I_0(\alpha \sqrt{1 - (\frac{2n}{N-1})^2})}{I_0(\alpha)} \quad \text{for } |n| \leq \frac{N-1}{2}$$

$$w_k[0] = 0, \text{ otherwise}$$

$\alpha \rightarrow$ adjustable parameter

min. stop band attenuation, min. pass band ripple $\rightarrow$ to design ideal filter.

$I_0(\alpha) \rightarrow$ modified zeroth order Bessel fn., of order 0.

$$I_0(\alpha) = 1 + \sum_{k=1}^{\infty} \left[\frac{1}{k!} \left(\frac{\alpha}{2} \right)^k \right]^2$$

$$= 1 + \frac{0.25 \alpha^2}{(1!)^2} + \frac{(0.25 \alpha^2)^2}{(2!)^2} + \dots$$

upto 25 terms.

* Design Kaiser Window.

passband ripple $\alpha_p = 20 \log \frac{1+\delta}{1-\delta}$

stopband attenuation $\alpha_s = -20 \log \delta$.

transition bandwidth in rad/s

$$B = \omega_s - \omega_p.$$

* Procedure:

1) Determine $h_d(n)$ using Fourier series method for an ideal freq response.

$$\omega_c = \frac{\omega_p + \omega_s}{2}$$

2) Find out α_p & α_s , choose

$$\delta = \min(\delta_1, \delta_2),$$

where $\delta_1 = 10^{-0.05\alpha_p'}$, $\delta_2 = \frac{10^{0.05\alpha_p'} - 1}{10^{0.05\alpha_p'} + 1}$

α_p' & α_s' are prescribed passband ripple & stopband attenuation.

3) Calculate α_s using

$$\alpha_s = -20 \log_{10} \delta$$

4) Determine ' α ' from foll. eqⁿs.

$$\alpha = 0 \text{ for } \alpha_s \leq 21$$

$$\alpha = 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) \quad \forall \quad 21 \leq \alpha_s \leq 50.$$

$$\alpha = 0.1102(\alpha_s - 8.7) \text{ for } \alpha_s > 50.$$

5) Choose parameter ' D '

$$D = 0.9222 \quad \forall \quad \alpha_s \leq 21$$

$$D = \frac{\alpha_s - 7.95}{14.36} \quad \forall \quad \alpha_s > 21$$

6) Choose filter order for lowest odd value of N

$$N \geq \frac{\omega_{sf} D}{B} + 1$$

ω_{sf} = Sampling freq in radian.

7> Compute window sequence using

$$w_k = \frac{I_0\left(\alpha \sqrt{1 - \left(\frac{2n}{N-1}\right)^2}\right)}{I_0(\alpha)} \quad \forall |n| \leq \frac{N-1}{2}$$

8> Compute modified impulse response using $h(n)$

$$h(n) = w_k[n] h_d[n]$$

$w_k[n]$ = coefficients found by Kaiser window

$h_d[n]$ = coefficients found by fourier series.

9> The transfer fn. is given as, $(N-1)/2$

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \left(h[0] + 2 \sum_{n=0}^{(N-1)/2} h(n)(z^n + z^{-n}) \right)$$

10> The magnitude response can be obtained by using

$$|H(e^{j\omega})| = \sum_{n=0}^{(N-1)/2} a(n) \cos \omega n$$

$$\text{where } a(0) = h\left[\frac{N-1}{2}\right]$$

$$a(n) = 2 h\left[\frac{N-1}{2} - n\right]$$

* for High Pass filter

$$B = \omega_p - \omega_s$$

$$H(e^{j\omega}) = \begin{cases} 0 & \forall |\omega| < \omega_c \\ 1 & \forall \omega_c \leq |\omega| \leq \frac{\omega_s f}{2} \end{cases}$$

$$\text{where } \omega_c = \frac{1}{2}(\omega_s + \omega_p)$$

* For bandpass filter

$$B = \min\left[(\omega_{p1} - \omega_{s1}), (\omega_{s2} - \omega_{p2})\right]$$

$$H(e^{j\omega}) = \begin{cases} 0 & \forall 0 \leq |\omega| \leq \omega_{c1} \\ & \omega_{c2} < |\omega| < \frac{\omega_s f}{2} \\ 1 & \forall \omega_{c1} \leq |\omega| \leq \omega_{c2} \end{cases}$$

$$= 1 \quad \forall \quad \omega_{c1} \leq |\omega| \leq \omega_{c2}$$

$$\omega_{c1} = \omega_{p1} - \frac{B}{2} \quad \omega_{c2} = \omega_{p2} + \frac{B}{2}$$

* Band Reject filter.

$$B = \min[(\omega_{s1} - \omega_{p1}), (\omega_{p2} - \omega_{s2})]$$

$$H(e^{j\omega}) = 1 \quad \forall \quad 0 \leq |\omega| \leq \omega_{c1} \quad \& \quad \omega_{c2} < |\omega| \leq \omega_{sf}/2$$

where

$$\omega_{c1} = \omega_{p1} + B/2, \quad \omega_{c2} = \omega_{p2} - B/2$$

freq. given in rad/sec, to convert it in radian

$$\omega_c \text{ (in radian)} = \omega_c T = \omega_c \frac{2\pi}{\omega_{sf}}$$

Q.) Design FIR low pass filter satisfying foll. specification.

$$\alpha_p \leq 0.1 \text{ dB}, \quad \alpha_s \geq 44.0 \text{ dB}, \quad \omega_p = 20 \text{ rad/sec}$$

$$\omega_s = 30 \text{ rad/sec}, \quad \omega_{sf} = 100 \text{ rad/sec}$$

→ from given specifications.

$$B = \omega_s - \omega_p = 30 - 20 = 10 \text{ rad/sec}$$

$$\omega_c = \frac{1}{2}(\omega_s + \omega_p) = 25 \text{ rad/sec}$$

$$\omega_c \text{ (in radian)} = \frac{25 \times 2\pi}{100} = \pi/2$$

$$H(e^{j\omega}) = 1 \quad \forall \quad |\omega| \leq \pi/2$$

$$= 0 \quad \forall \quad \pi/2 < |\omega| < 2\pi$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega$$

$$= \frac{\sin(\pi/2)n}{\pi n}$$

$$\delta_1 = 10^{-0.05(44)} = 6.3096 \times 10^{-3}$$

$$\delta_2 = \frac{10^{0.05(0.1)}}{10^{0.05(0.1)} + 1}$$

$$= 5.7563 \times 10^{-3}$$

$$\delta = \min(\delta_1, \delta_2)$$

$$\therefore \delta = 5.7563 \times 10^{-3}$$

$$\alpha_s = -20 \log \delta = 44.797 \text{ dB}$$

$$\alpha = 0.5842(\alpha_s - 21)^{0.4} + 0.8886(\alpha_s - 21)$$

$$= 3.9524$$

$$D = \frac{\alpha_s - 7.95}{14.36} = 2.566$$

$$N \geq \frac{W_s D}{B} + 1$$

$$\geq \frac{100(2.566)}{10} + 1$$

$$\geq 26.66$$

$$w_k(n) = \frac{I_0 \left[\alpha \sqrt{1 - \left(\frac{2n}{N-1} \right)^2} \right]}{I_0(\alpha)} \quad N=27$$

$$\alpha = 3.9524$$

$$n=0, w_k[0] = \frac{I_0[3.9524 \sqrt{1-0}]}{I_0[3.9524]} = 1$$

$$w_k(1) = w_k(-1) = \frac{I_0 \left[3.9524 \sqrt{1 - \left(\frac{2}{27-1} \right)^2} \right]}{I_0[3.9524]}$$

$$= \frac{I_0[3.9524 \sqrt{0.997}]}{I_0[3.9524]} = \frac{10.8464}{10.8464} \frac{I_0[3.9405]}{I_0[3.9524]}$$

$$= 0.9824$$

$$I_0(3.9524) = 1 + \frac{0.25(3.9524)^2}{(1!)^2}$$

$$+ \frac{0.25(3.9524)^2}{(2!)^2} + \dots$$

$$= 10.8464$$

$$w_k(2) = w_k(-2) = 0.9603$$

$$w_k(3) = w_k(-3) = 0.9124$$

$$h[n] = h_d[n] w_k[n]$$

20/11/20

Module 4

* IIR (Impulse Invariance filter)
Infinite Impulse Response.

- 17 Impulse invariance method.
- 27 Bilinear Transformation method (BLT)
- 37 Matched Z transform method.
- 47 Approximation of derivation method.

* Impulse invariance method.

$$h[nT_s] = h(t) \Big|_{t=nT_s}$$

where $h(t)$ is analog impulse response & T_s is the sampling period.

The mapping points b/w z plane & s plane is by means of sampling process.

We can write relationship b/w z transform of $h(n)$ & laplace transform of $h(t)$.

$$H[z] \Big|_{z=e^{sT_s}} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} H(s + j\frac{2\pi k}{T_s})$$

where $s = \sigma + j\omega$ &

$$H[z] \Big|_{z=e^{sT_s}} = \sum_{n=-\infty}^{\infty} h(n) e^{-sT_s n}$$

Now consider the mapping of points from s plane to z plane as $z = e^{(s + j\omega)T_s}$

$$z = e^{sT_s} = e^{\sigma T_s} \cdot e^{j\omega T_s}$$

$$z = e^{\sigma T_s} \cdot e^{j\omega T_s}$$

Now representing 'z' in polar form, we get

$$z = r e^{j\omega}$$

$$r \cdot e^{j\omega} = e^{\sigma T_s} \cdot e^{j\omega T_s}$$

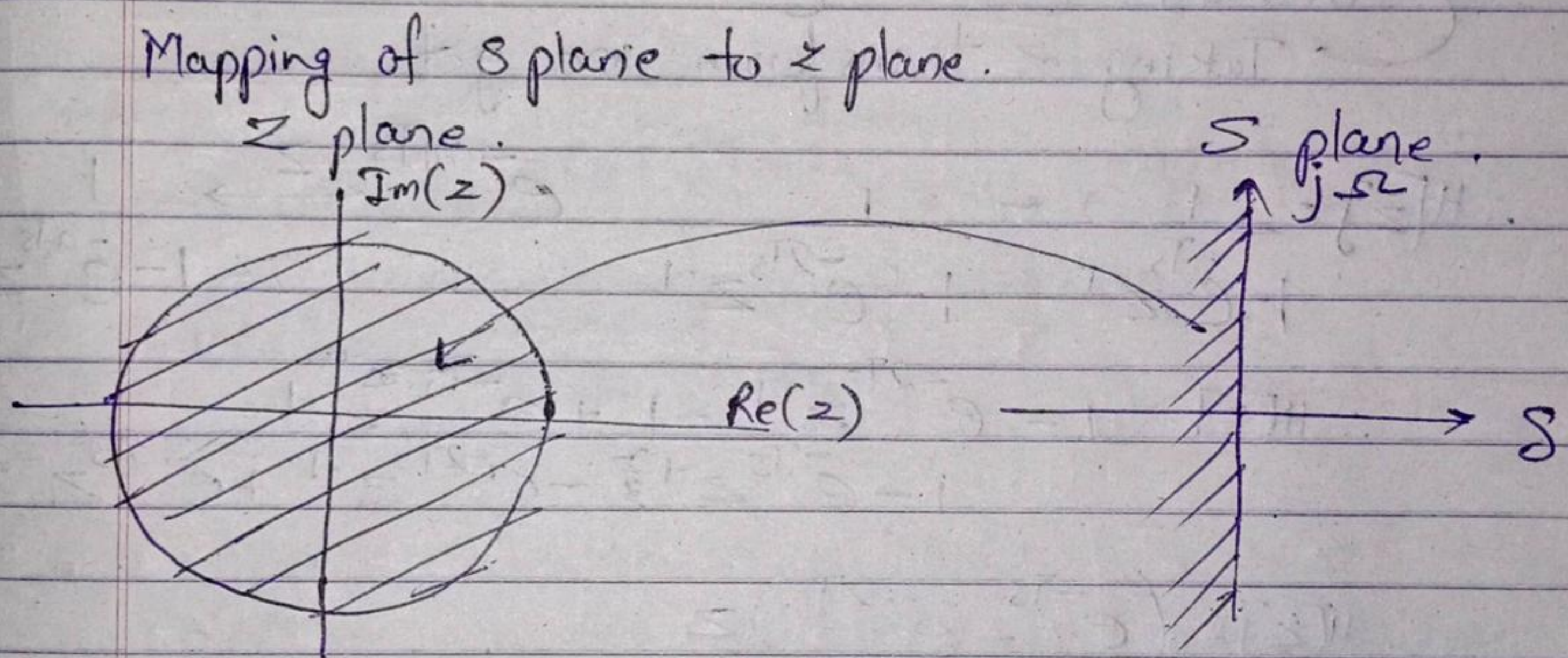
$$r = e^{\delta T_s}, \quad e^{j\omega} = e^{j\Omega T_s}$$

$$\boxed{\omega = \Omega T_s}$$

- 1) if $\delta = 0$, then $r = 1$
- 2) if $\delta < 0$, r is in b/w 0 & 1
- 3) if $\delta > 0$, then $r > 1$.
- 4) if analog frequency, $\Omega = \frac{\pi}{T_s}$ then

$$\text{digital frequency } \omega = \frac{\pi}{T_s} \times T_s = \pi$$

- 5) if $\Omega = -\frac{\pi}{T_s}$, then $\omega = \frac{-\pi}{T_s} \times T_s = -\pi$



$$\Omega \rightarrow \frac{\pi}{T_s} \text{ to } \frac{-\pi}{T_s}, \quad \omega = -\pi \text{ to } +\pi$$

Here, aliasing effect is not.

Q.] Transfer fn. of analog filter is given as

$$H(s) = \frac{1}{(s+1)(s+2)}$$

1) Find $H(z)$ using impulse invariance method.

2) Find $H(z)$ if sampling freq F_s is 5 samples/sec.

$$\rightarrow \frac{H(s)}{1} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A(s+2) + B(s+1) = 1$$

$$sA + 2A + Bs + B = 1$$

$$2A + B = 1$$

$$A + B = 0$$

$$2A - A = 1$$

$$A = -B$$

$$A = 1$$

$$B = -1$$

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Taking inverse Laplace transform

$$h(t) = e^{-t} - e^{-2t}; \quad t > 0.$$

Sampling $h(t)$

$$h[nT_s] = h(t) \Big|_{t=nT_s} \quad \text{Ts = sampling period.}$$

$$h[nT_s] = e^{-nT_s} - e^{-2nT_s}$$

→ Taking z transform, we get

$$\therefore H[z] = \frac{1}{1 - e^{-T_s} z^{-1}} - \frac{1}{1 - e^{-2T_s} z^{-1}} \quad e^{-nT_s} \xrightarrow{z} \frac{1}{1 - e^{-nT_s} z^{-1}}$$

$$\therefore H[z] = \frac{1 - e^{-2T_s} z^{-1} - 1 + e^{-T_s} z^{-1}}{1 - e^{-T_s} z^{-1} - e^{-2T_s} z^{-1} + e^{-3T_s} z^{-2}}$$

$$\therefore H[z] = \frac{(e^{-T_s} - e^{-2T_s})z}{z^2 - z(e^{-T_s} + e^{-2T_s}) + e^{-3T_s}}$$

$$F_s = 5 \text{ sps} \quad T_s = \frac{1}{5} = 0.2 \text{ sec.}$$

$$\therefore H[z] = \frac{0.148z}{z^2 - 1.489z + 0.5488}$$

Q.2] Convert the analog filter with system transfer fn. $H(s) = \frac{s+0.1}{(s+0.1)^2 + 9}$ into digital IIR filter using impulse invariance method.

$$\rightarrow H(s) = \frac{s+0.1}{(s+0.1)^2 + 9}$$

$$H(s) = \frac{s+0.1}{(s+0.1-j3)(s+0.1+j3)}$$

$$= \frac{A}{s+0.1-j3} + \frac{B}{s+0.1+j3}$$

$$A = 1/2, \quad B = 1/2$$

$$H(s) = \frac{1/2}{s+0.1-j3} + \frac{1/2}{s+0.1+j3}$$

Taking inverse Laplace transform

$$h(t) = \frac{1}{2} e^{-(0.1-j3)t} + \frac{1}{2} e^{-(0.1+j3)t}; \quad t > 0$$

sampling $h(t)$ i.e. $t = nT_s$

$$h(nT_s) = \frac{1}{2} e^{-(0.1-j3)nT_s} + \frac{1}{2} e^{-0.1nT_s} \cdot e^{-j3nT_s}$$

Take z transform of above eqⁿ.

$$H(z) = \frac{\frac{1}{2}}{1 - e^{-0.1T_s} \cdot e^{j3T_s} z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-0.1T_s} \cdot e^{-j3T_s} z^{-1}}$$

* Bilinear Transformation Method (BLT)

Let us consider, the first order linear filter system function $H(s)$

$$H(s) = \frac{b}{s+a} \quad \text{--- (1)}$$

The system fn. is equivalent to digital filter $H[z]$

$$H[z] = \frac{b}{\frac{2}{T} \left[\frac{1-z^{-1}}{1+z^{-1}} \right] + a} \quad \text{--- (2)}$$

T/T_s

$$\left[\frac{1-z^{-1}}{1+z^{-1}} \right] + a$$

$T/T_s = \text{Sampling period}$

Mapping from s plane to z plane is given as

$$s = \frac{2}{T} \left[\frac{1-z^{-1}}{1+z^{-1}} \right] \quad \text{or} \quad z = \frac{s \cdot T/2 + 1}{s \cdot T/2 - 1} \quad \text{--- (3)}$$

★ The mapping of s plane to z plane by relation $s = \frac{2}{T} \left[\frac{1-z^{-1}}{1+z^{-1}} \right]$ is called as Bilinear transformation characteristics.

Let $z = re^{j\omega}$ & $s = \sigma + j\Omega$

eqn (3) can be written as

$$\begin{aligned} \sigma + j\Omega &= \frac{2}{T} \left[\frac{1 - (re^{j\omega})^{-1}}{1 + (re^{j\omega})^{-1}} \right] \quad \text{put values of } z \text{ \& } s. \\ &= \frac{2}{T} \left[\frac{(re^{j\omega}) - 1}{(re^{j\omega}) + 1} \right] \times \frac{re^{j\omega} + 1}{re^{j\omega} + 1} \\ &= \frac{2}{T} \left[\frac{r^2 - re^{-j\omega} + re^{j\omega} - 1}{r^2 + re^{j\omega} + re^{j\omega} + 1} \right] \\ &= \frac{2}{T} \left[\frac{r^2 - 1 + j2r \left(\frac{e^{j\omega} - e^{-j\omega}}{2j} \right)}{r^2 + 2r \left(\frac{e^{j\omega} + e^{-j\omega}}{2} \right) + 1} \right] \\ &= \frac{2}{T} \left[\frac{r^2 - 1}{r^2 + 2r \cos \omega + 1} + j \frac{2r \sin \omega}{r^2 + 2r \cos \omega + 1} \right] \end{aligned}$$

Equating real & imaginary parts

$$\sigma = \frac{2}{T} \left[\frac{r^2 - 1}{r^2 + 2r \cos \omega + 1} \right] \quad \& \quad \Omega = \frac{2}{T} \left[\frac{2r \sin \omega}{r^2 + 2r \cos \omega + 1} \right]$$

$$\text{If } r < 1, \quad \sigma < 0$$

$$r > 1, \quad \sigma > 0$$

$$r = 1, \quad \sigma = 0$$

For $r=1$, value of analog frequency Ω becomes,

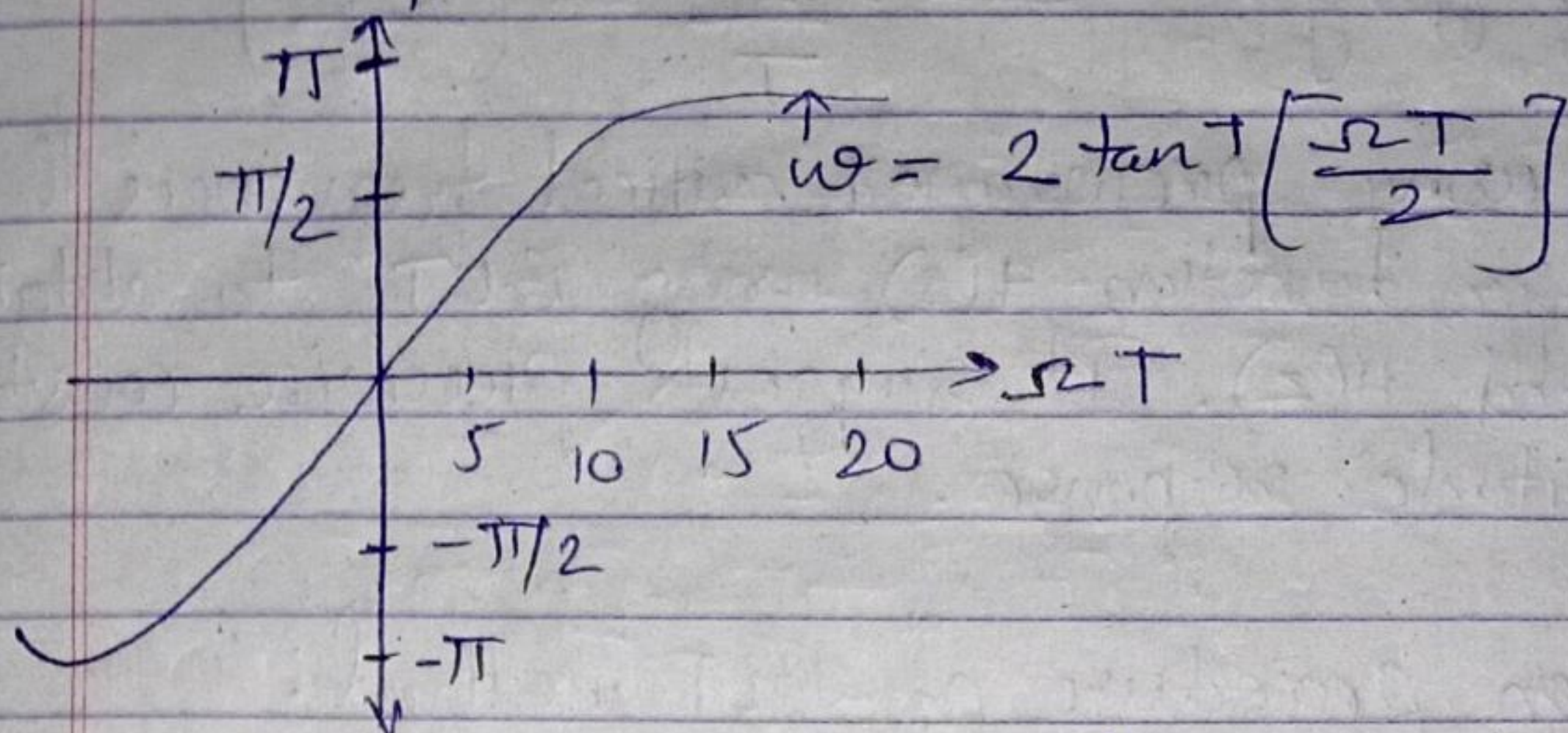
$$\begin{aligned} \Omega &= \frac{2}{T} \left[\frac{2(1)(\sin \omega)}{(1)^2 + 2(1)(\cos \omega) + 1} \right] \\ &= \frac{2}{T} \frac{\sin \omega}{1 + \cos \omega} \end{aligned}$$

Referring to trigonometrical identities,

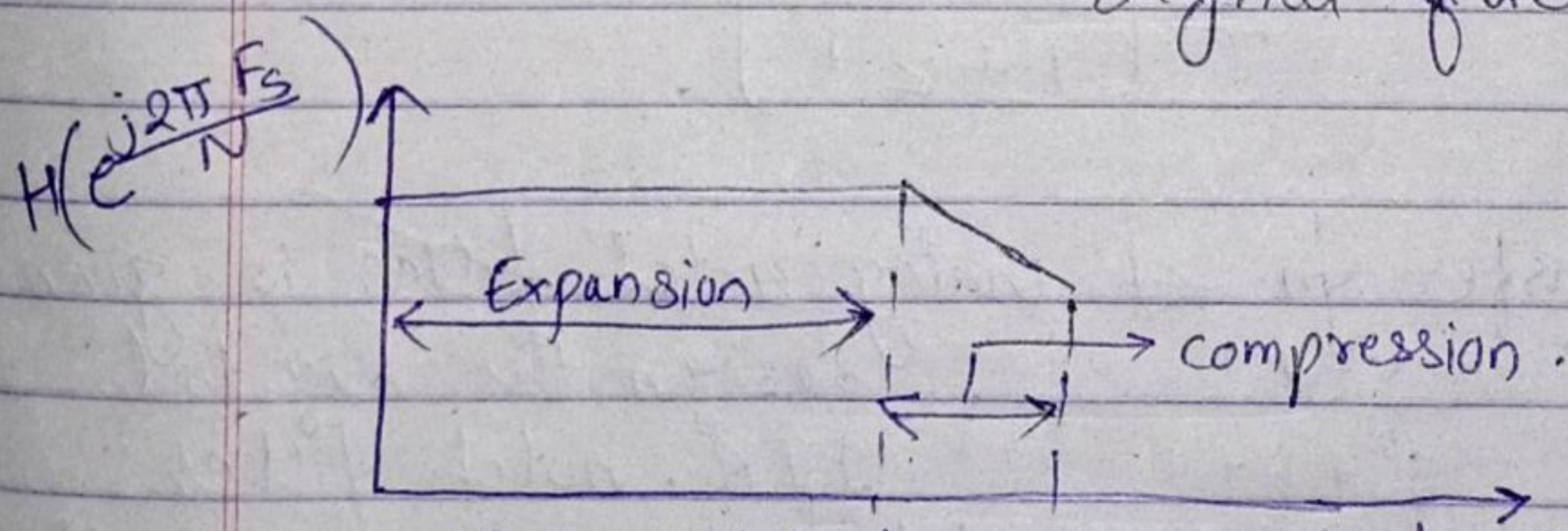
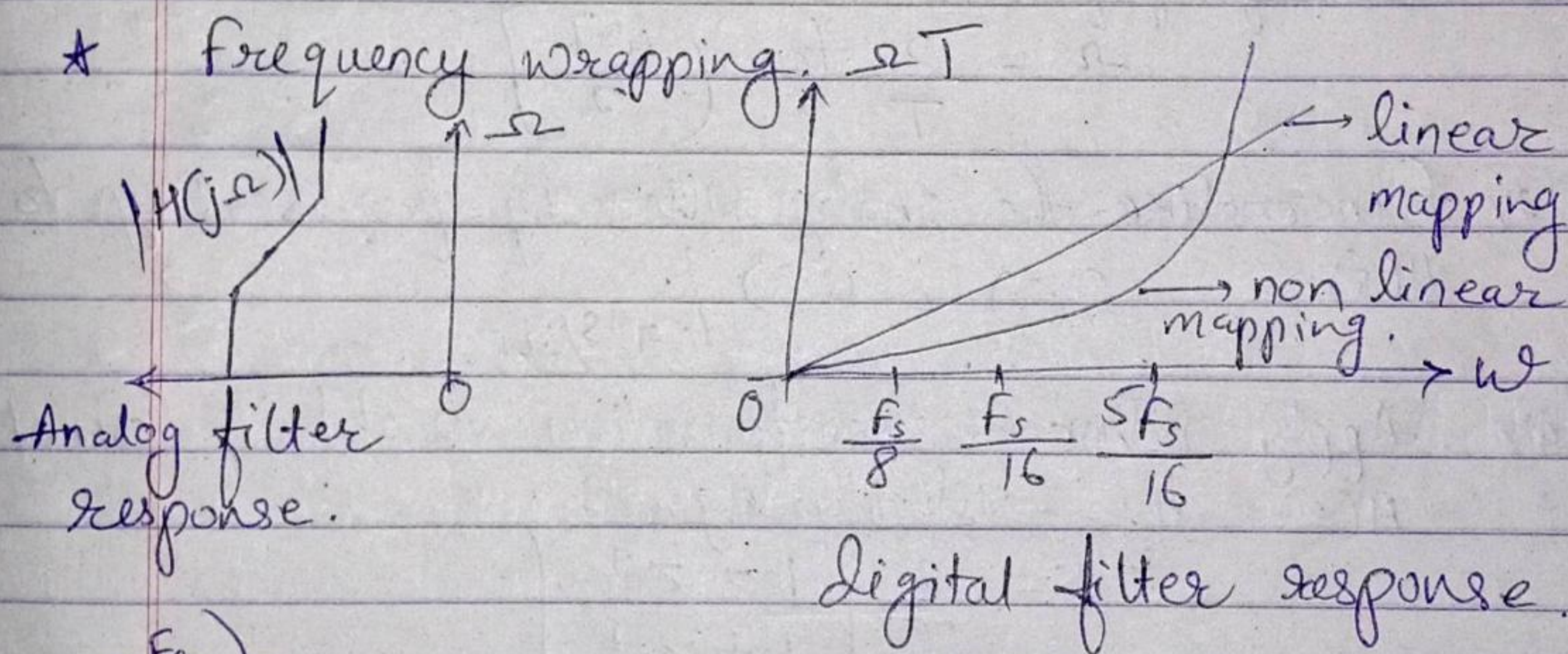
$$2 \cdot \frac{\sin \omega}{2} \cdot \frac{\cos \omega}{2} = \sin \omega$$

$$2 \cos^2 \frac{\omega}{2} = 1 + \cos \omega, \text{ we get}$$

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2} \quad \text{or} \quad \omega = 2 \tan^{-1} \left(\frac{\Omega T}{2} \right)$$



There is a tangent fn. present, due to non linearity of tangent fn. there exist frequency compression. This is also called as frequency wrapping.



The amplitude response of digital IIR filter is expanded at lower frequency & compressed at higher frequency. This occurs due to non linear mapping in BLT.

In order to avoid this wrapping, we go for pre wrapping.

★ Pre wrapping: $\Omega = \frac{2}{T} \tan\left(\frac{\omega T}{2}\right)$

$H(s)$ using prewrapped critical frequencies then transfer function $H(s)$ using BLT to obtain desired $H(z)$. It suggests piecewise constant magnitude response.

★ Design procedure of BLT method:

- 1) Determine the normalized form of given transfer fn. $H(s)$
- 2) Pre wrap the critical digital freq. ω_c to obtain an analog freq Ω .

$$\Omega = \frac{2}{T} \tan\left(\frac{\omega T}{2}\right)$$

- 3) Denormalize the analog filter by freq. scaling to $H(s)$

$$\hat{H}(s) = H(s) \Big|_{s=s/\Omega}$$

- 4) Apply Bilinear transformation to obtain desired $H(z)$ by substituting.

$$s = \frac{2}{T} \left[\frac{1-z^{-1}}{1+z^{-1}} \right]$$

Q.1] The transfer fn. of analog notch filter is given as $H(s) = \frac{s^2+1}{s^2+s+1}$. design the digital IIR notch filter using BLT with notch frequency F_z 60 Hz & sampling freq 960 sps.

→ Cut off freq of digital filter is given as,
 $\omega_c = 2\pi f_c = 2\pi \times 60 = 120\pi \text{ rad/sec}$

$$2) \quad \Omega = \tan\left[\frac{\omega T}{2}\right] = \tan\left[\frac{120\pi (1)}{2 \times 960}\right] = 0.1989$$

3) Freq. scaling of $H(s)$ becomes,

$$\hat{H}(s) = H(s) \Big|_{s = s/\Omega = s/0.1989}$$

$$\hat{H}(s) = \frac{(s/0.1989)^2 + 1}{(s/0.1989)^2 + (s/0.1989) + 1}$$

4) Applying BLT to get $H(z)$

$$H(z) = \hat{H}(s) \Big|_{s = \frac{z-1}{z+1}}$$

$$= \frac{s^2 + 0.0396}{s^2 + 0.1989s + 0.0396}$$

$$H(z) = \frac{1.0396 - 1.9208z^{-1} + 1.0396z^{-2}}{1.2385 - 1.9208z^{-1} + 0.8407z^{-2}}$$

Q.2] The transfer fn. of analog LPF is given as, $H(s) = 1/s+1$
 Bandwidth of LPF = 1 rad/sec with the help of
 BLT, design digital LPF with bandwidth 20 Hz
 at sampling freq. of 60 sps.

→ The cutoff freq. of digital IIR filter is
 given as $\omega_c = 2\pi f_c = 2\pi \times 20 = 40\pi \text{ rad/sec}$.

$$2) \quad \Omega = \frac{2}{T} \tan\left[\frac{\omega T}{2}\right] = 2 \times 60 \tan\left[\frac{40\pi}{2(60)}\right] =$$

$$= 120 \tan\left(\pi/3\right) = 120\sqrt{3}$$

3) Frequency scaling $H(s)$ becomes,

$$\hat{H}(s) = H(s) \Big|_{s = s/\Omega = s/120\sqrt{3}}$$

$$\hat{H}(s) = \frac{1}{\frac{s}{120\sqrt{3}} + 1} = \frac{120\sqrt{3}}{s + 120\sqrt{3}}$$

47 Applying BLT

$$H[z] = \frac{120\sqrt{3}}{\frac{2}{T} \left[\frac{z-1}{z+1} \right] + 120\sqrt{3}}$$

$$= \frac{120\sqrt{3} (z+1)}{120[z-1] + [z+1] \times 120\sqrt{3}}$$

* Low pass digital Butterworth filter -

N (order of filter)	Transfer fn.	
	For LPF	For HPF
1	$\frac{1}{s+1}$	$\frac{s}{s+1}$
2	$\frac{1}{s^2 + \sqrt{2}s + 1}$	$\frac{s^2}{s^2 + \sqrt{2}s + 1}$
3	$\frac{1}{s^3 + 2s^2 + 2s + 1}$	$\frac{s^3}{s^3 + 2s^2 + 2s + 1}$
4	$\frac{1}{s^4 + 2.613s^3 + 3.412s^2 + 2.613s + 1}$	$\frac{s^4}{s^4 + 2.613s^3 + 3.412s^2 + 2.613s + 1}$

If we ignore z^{-1} term in Ω_p calculation also ignore it while calculating $H(z)$

(*) Design the second order low pass digital butterworth filter whose cutoff frequency is 1 kHz at sampling freq. 10^4 sps. Use BLT method.

→ Given that

$$\text{cutoff freq} = 1 \text{ kHz}, f_s = 10^4 \text{ sps.}$$

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

cutoff freq. for digital filter

$$\omega_c = 2 \tan^{-1} \left[\frac{\omega_c T_s}{2} \right] \quad \omega_c = 2\pi f_c = 2\pi \times 10^3$$

$$= 6.28 \text{ rad/sec}$$

$$\Omega_p = \tan \left[\frac{\omega_c T_s}{2} \right]$$

$$= \tan \left[\frac{2000\pi}{10 \times 1000 \times 2} \right] = \tan \left(\frac{\pi}{10} \right) = 0.324$$

frequency scaling of $H(s)$ becomes.

$$\hat{H}(s) = H(s) \Big|_{s = s/\Omega_p} = \frac{s}{0.324}$$

$$\therefore \hat{H}(s) = \frac{1}{\left(\frac{s}{0.324} \right)^2 + \sqrt{2} \frac{s}{0.324} + 1}$$

$$= \frac{0.324 \times 0.324}{s^2 + 0.4596s + 0.1056}$$

Apply BLT

$$H(z) = \hat{H}(s) \Big|_{s = \frac{z-1}{z+1}}$$

$$\hat{H}(s) = \frac{0.1056}{\left(\frac{z-1}{z+1} \right)^2 + 0.4596 \left(\frac{z-1}{z+1} \right) + 0.1056}$$

$$= \frac{0.1056(z+1)^2}{(z-1)^2 + 0.4596(z+1)(z-1) + 0.1056(z+1)^2}$$

$$H(z) = \frac{0.0676 (1 + 2z^{-1} + z^{-2})}{1 - 1.1422z^{-1} + 0.4124z^{-2}}$$

Q] Design a second order, digital Butterworth band pass filter with following specifications.
 lower cutoff freq = 210 Hz, upper cutoff freq = 330 Hz
 sampling freq = 960 sps.

Std. transfer fn.

$$H(s) = \frac{\omega s}{s^2 + \omega s + \omega_0^2} \quad \omega = \text{filter band width}$$

ω_0 = centre frequency.

cutoff freq. for digital filter are

$$\omega_{c1} = 2\pi f_{LC}, \quad \omega_{c2} = 2\pi f_{HC}$$

$$= 2\pi \times 210$$

$$= 420\pi \text{ rad/sec.}$$

$$= 2\pi \times 330$$

$$= 660\pi \text{ rad/sec.}$$

$$\Omega_{p1} = \tan\left[\frac{\omega_{c1} T_s}{2}\right] = \tan\left[\frac{420\pi}{2 \times 960}\right] = 0.8206$$

$$\Omega_{p2} = \tan\left[\frac{\omega_{c2} T_s}{2}\right] = \tan\left[\frac{660\pi}{2 \times 960}\right] = 1.8708$$

$$\text{Bandwidth of filter, } \omega = \Omega_{p2} - \Omega_{p1} = 1.8708 - 0.8206$$

$$\text{centre freq. } \Omega_0^2 = \Omega_{p1} \times \Omega_{p2}$$

$$\Omega_0^2 = 1.535$$

$$H(s) = \frac{1.05s}{s^2 + 1.05s + 1.535}$$

$$H(z) = H(s) \Big|_{s = \frac{z-1}{z+1}}$$

$$= \frac{1.05 \left(\frac{z-1}{z+1}\right)}{\left(\frac{z-1}{z+1}\right)^2 + 1.05 \left(\frac{z-1}{z+1}\right) + 1.535}$$

For designing LP, HP, Band pass & Band stop filter from low pass filters, replace the value of s in transfer fn. $H(s)$ of LPF by

$$s = \frac{s}{\omega_p}$$

$$s = \omega_p/s \quad \text{for LPF to HPF.}$$

$$s = \frac{s^2 + \omega_0^2}{\omega_s} \quad \text{for LPF to Band pass filter}$$

$$s = \frac{\omega_s s}{s^2 - \omega_0^2} \quad \text{for LPF to Band stop filter}$$

$$\omega_0^2 = \omega_1 \omega_2$$

$$\omega = \omega_2 - \omega_1$$

Q Design first order high pass filter for cutoff freq. 30 Hz . Sampling freq. $= 150 \text{ Hz}$.

$$\rightarrow f_c = 30 \text{ Hz}, \quad f_s = 150 \text{ Hz}$$

$$1) H_L(s) = \frac{1}{s+1}$$

2) cutoff freq. of digital freq. ω_c .

$$\omega_c = 2\pi f_c = 2 \times \pi \times 30 \text{ Hz} = 60\pi \text{ Hz.}$$

$$3) \omega_p = \tan \frac{\omega_c T}{2} = \tan \frac{60\pi}{2 \times 150} = \tan \frac{\pi}{5}$$

$$= 0.7265$$

4) To convert from low pass to high pass filter.

$$H_H(s) = H_L(s) \Big|_{s = \frac{\omega_p}{s}}$$

$$H_H(s) = \frac{1}{\frac{\omega_p}{s} + 1} = \frac{s}{\omega_p + s} = \frac{s}{0.7265 + s}$$

5) Apply BLT

$$H(z) = H(s) \Big|_{s = \frac{z-1}{z+1}}$$

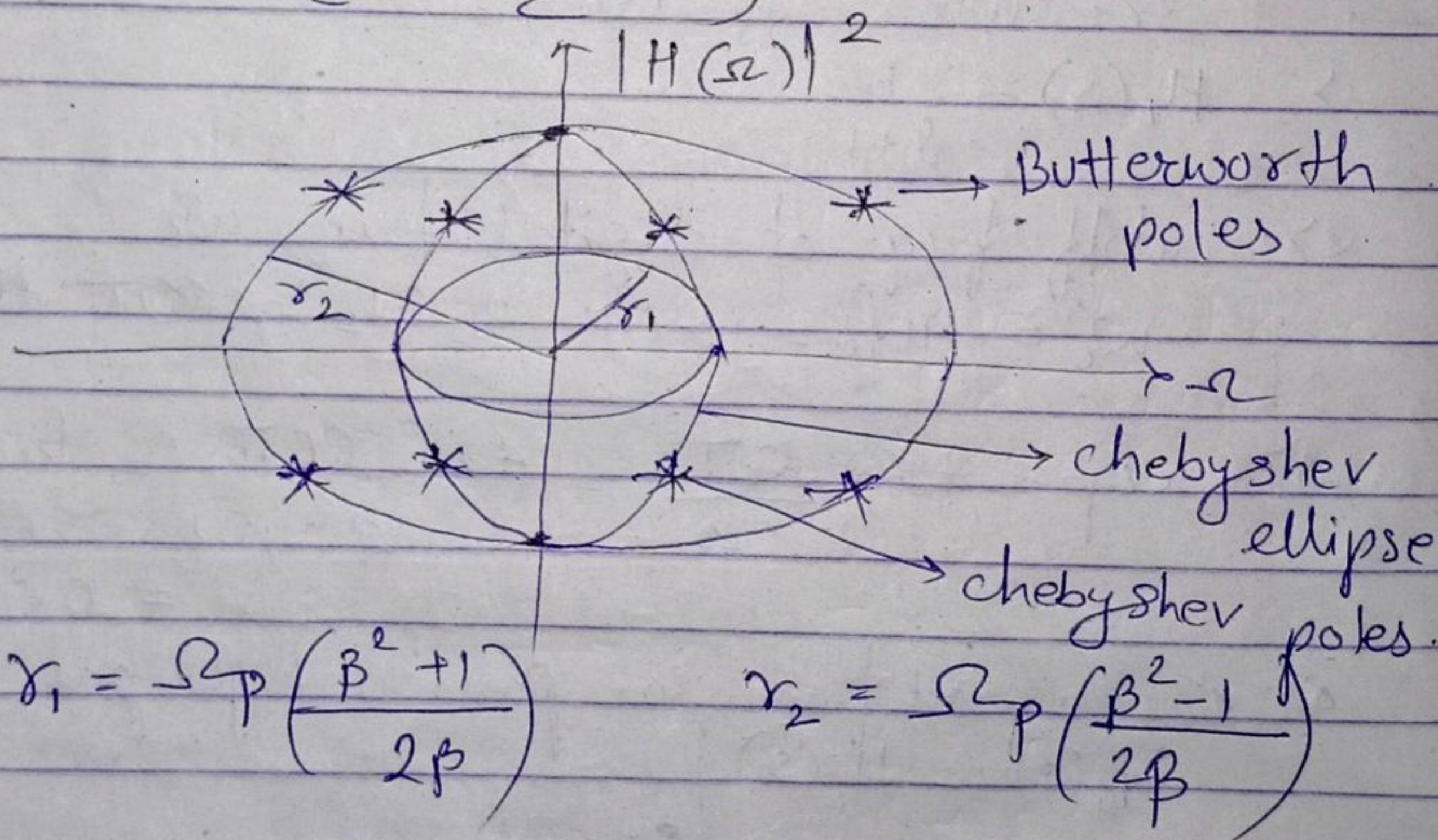
$$= \frac{z-1}{(z+1) + 0.7265(z+1)} = \frac{z-1}{1.7265z - 0.2735}$$

$$\therefore H[z] = \frac{1-z^{-1}}{-0.2735z^{-1} + 1.7265}$$

- * Poles of Type I chebyshev filter. Poles of Butterworth filter lies on circle whereas poles of chebyshev filter lies on ellipse in s plane by j ω axis.

Let us define the parameter β in terms of ϵ (filter parameter related to passband)

$$\beta = \left(\frac{1 + \sqrt{1 + \epsilon^2}}{\epsilon} \right)^{1/N}$$



- * Angular position of the Butterworth filter is given as

$$\theta_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \quad k=0, 1, 2, \dots, (N-1)$$

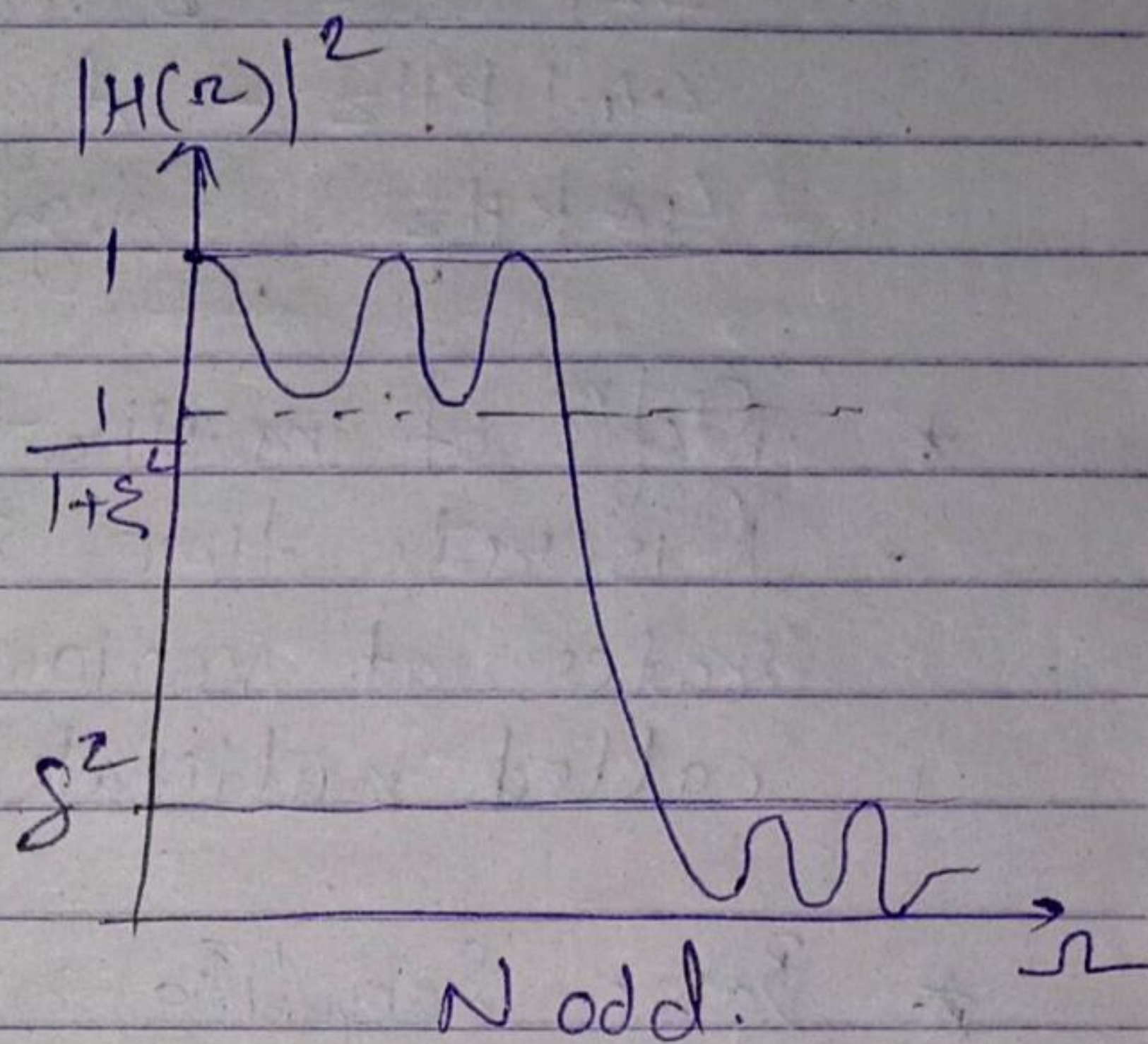
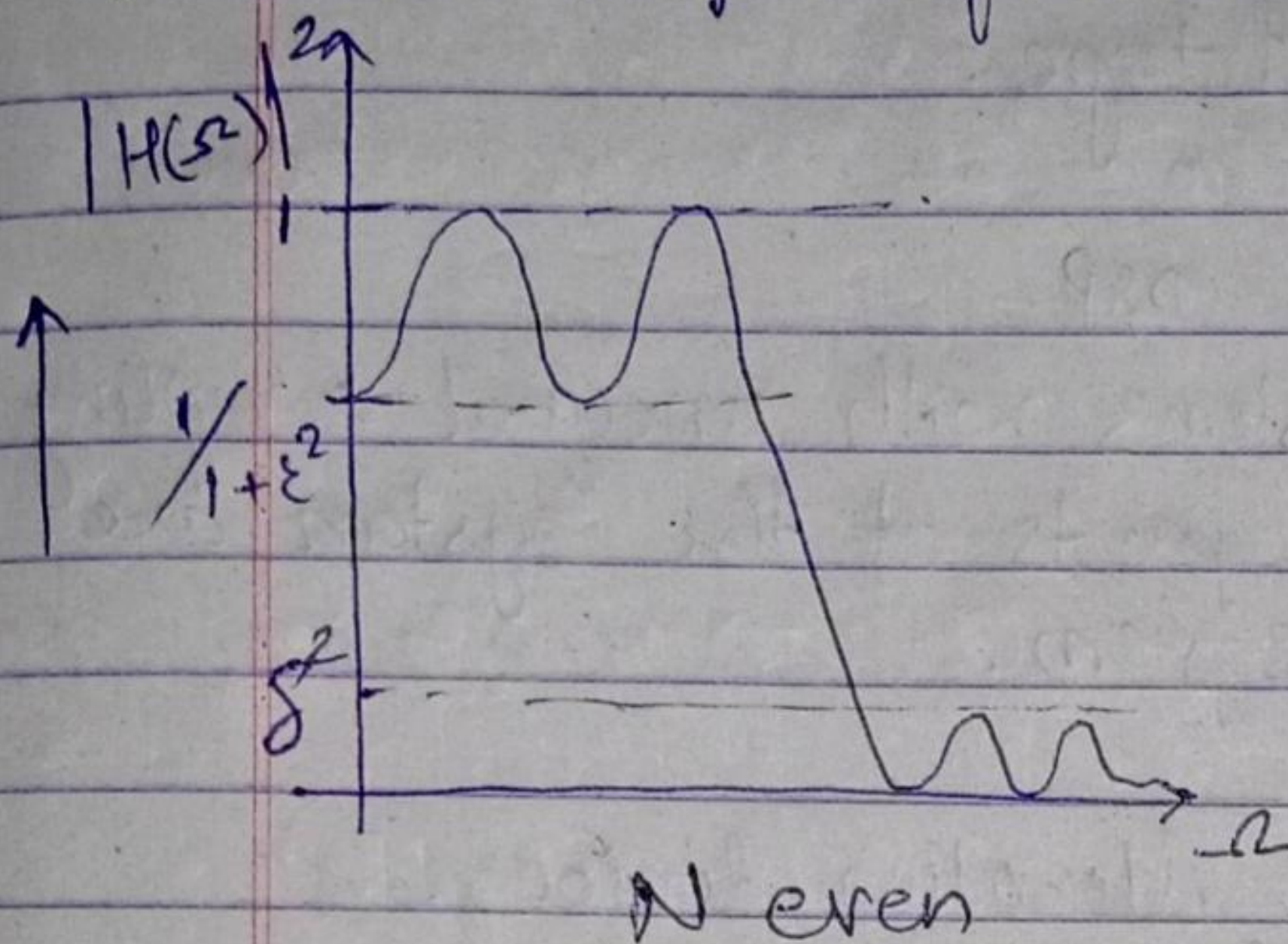
* Position of poles for chebyshev filter lie on ellipse with coordinates (x_k, y_k) for $k=0, 1, \dots, (N-1)$

$$x_k = \sigma_2 \cos \theta_k, \quad k=0, 1, 2, \dots, (N-1)$$

$$y_k = \sigma_1 \sin \theta_k, \quad k=0, 1, 2, \dots, (N-1)$$

Pole position: $\sigma_1 \cos \theta_k + j \sigma_2 \sin \theta_k$

* For elliptic filters.



$$|H(z)|^2 = \frac{1}{1 + \varepsilon^2 U_N^2(z/\Omega_p)}$$

$U_N(x)$ = Jacobian elliptic fn. of order N

ε = pass band related parameter

Ω_p = passband edge freq.

Module 5

★ Multirate DSP

- Earlier discussed DSP structures are single rate system.
- Sampling rate same @ I/P, O/P ~~at~~ intermediate nodes.

Digital Audio.

- 32 KHz - broadcasting.
- 44.1 KHz - digital audio.
- 48 KHz - digital tape.

★ Defⁿ of multirate DSP.

- Discrete time systems with unequal sampling rates at various parts of the system are called multirate system.

★ Basic Sampling rate alteration devices.

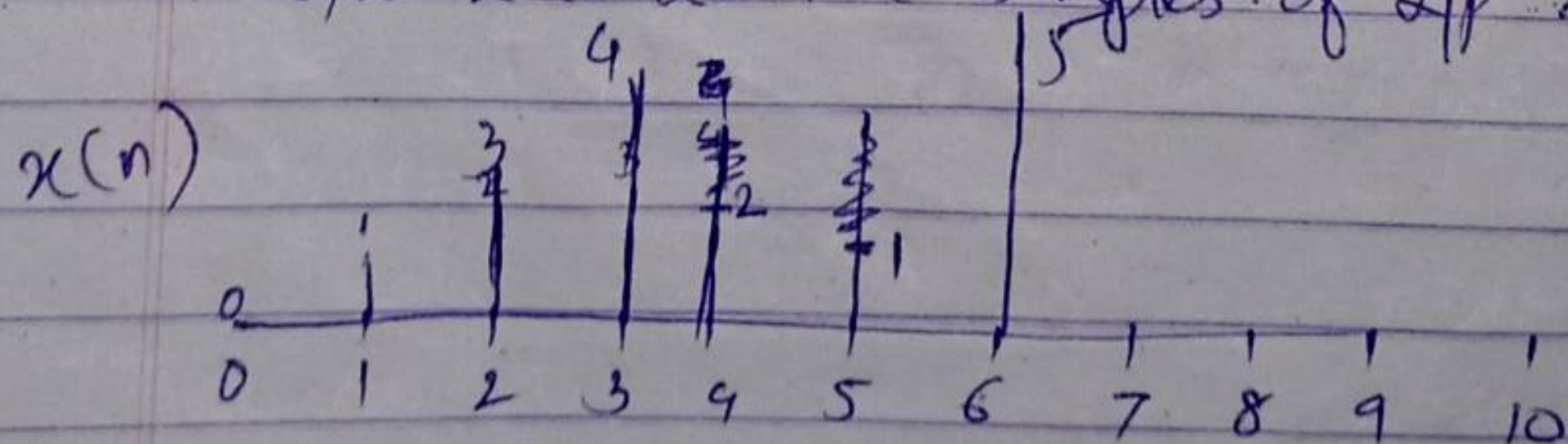
- 1) Upsampler
- 2) Down sampler

An up sampler with upsampling factor 'L' where L is +ve integer, develops output sequence $x_u[n]$ with sampling rate i.e. 'L' times larger than that of I/P sequence.

$$x(n) \rightarrow \boxed{\uparrow L} \rightarrow x_u[n]$$

Sampling rate expander or expander.

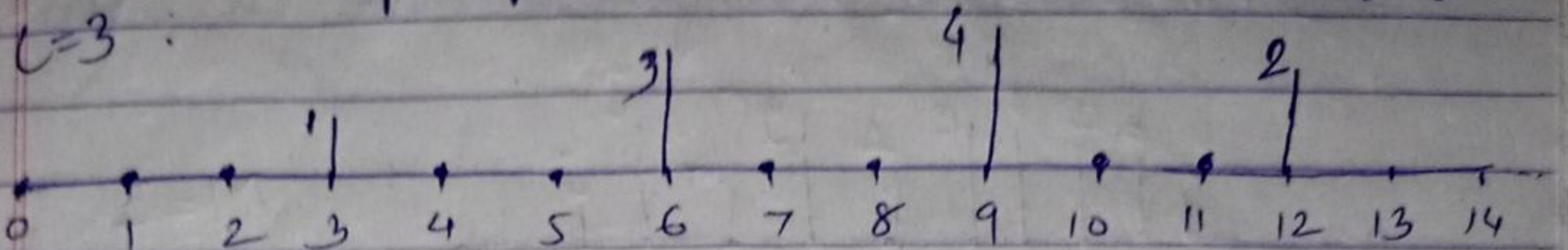
- ★ Upsampling operation is implemented by inserting (L-1) equidistant zeros (zero valued samples) b/w 2 consecutive samples of I/P sequence.



$$L=3$$

(L-1) = 2 $\rightarrow$ insert zero valued samples.

upsampled sequence with upsampling factor



$$x_u[n] = \begin{cases} x[n/L] & , n=0, \pm L, \pm 2L, \dots \\ 0 & , \text{otherwise} \end{cases}$$

Let $L=3$

$$\begin{aligned} x_u[0] &= x[0/3] = 0 \\ x_u[1] &= x[1/3] = 0 \\ x_u[2] &= x[2/3] = 0 \\ x_u[3] &= x[3/3] = 1 \\ x_u[4] &= x[4/3] = 0 \end{aligned}$$

In practice the zero valued samples inserted by upsampler are interpolated using some type of filtering process. in order that the new higher rate sequence has no unnecessary spectral components. So this upsampler is also called as interpolator.

★ Down sampling:

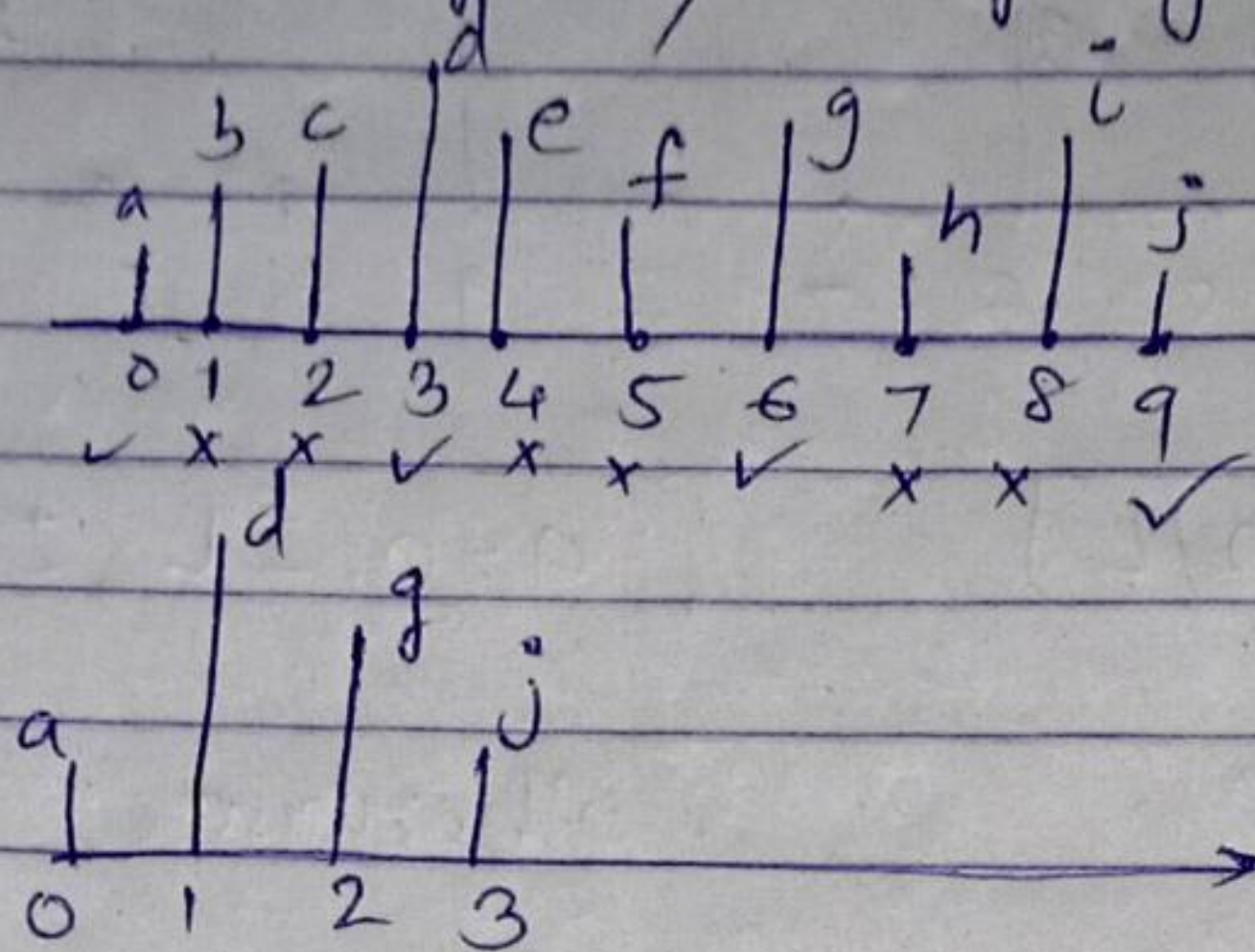
A down sampler with down sampling rate factor M , where M is +ve integer creates an output sequence $y[n]$ with sampling rate that is $(1/M)^{th}$ of that I/P sequence.

down sampling operation is implemented by removing $(M-1)$ in between samples. to generate o/p

$$y[n] = x[nM]$$

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow y[n]$$

Down sampler / sampling rate compressor



$$M=3$$

$$M-1=2$$

As a result all I/P samples with indices equal to integer multiple of M are retained & all others are discarded.

Down sampler

$$x(n) = x_a(nT) \rightarrow \boxed{\downarrow M} \rightarrow y(n) = x_a(nMT)$$

I/P sampling frequency $F_T = \frac{1}{T}$

O/P sampling frequency $f_T = \frac{F_T}{M} = \frac{1}{MT}$

Up sampler

$$x[n] = x_a\left[\frac{nT}{L}\right] \rightarrow \boxed{\uparrow L} \rightarrow x_u[n] = x_a\left(\frac{nT}{L}\right),$$

I/P sampling frequency $F_T = \frac{1}{T}$

$$\forall n=0, \pm L, \pm 2L, \dots$$

0, otherwise

O/P sampling frequency

$$F_T' = LF_T = \frac{L}{T}$$

* Perfect reconstruction multirate system.

A multirate structure in which the o/p sequence is delayed & scaled replica of I/P sequence is called as perfect reconstruction multirate system.

★ Frequency domain representation of upsampler

$$x_u[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases}$$

if $L = 3$

$$\begin{aligned} x_u[0] &= x\left[\frac{0}{3}\right] = x(0) \\ x_u[3] &= x\left[\frac{3}{3}\right] = x(1) \\ x_u[6] &= x\left[\frac{6}{3}\right] = x(2) \\ x_u[9] &= x\left[\frac{9}{3}\right] = x(3) \end{aligned}$$

For upsampling factor of 2

$$x_u[n] = \begin{cases} x\left[\frac{n}{2}\right], & n = 0, \pm 2, \pm 4, \dots \\ 0, & \text{otherwise} \end{cases}$$

In terms of Z transform, the I/P. o/p relationship is given as,

$$X_u[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x\left[\frac{n}{2}\right] z^{-n}$$

put $n/2 = m \therefore n = 2m$

$$\therefore X_u[z] = \sum_{m=-\infty}^{\infty} x[m] z^{-2m}$$

for 'n' even

$$X_u[z] = X[z^2]$$

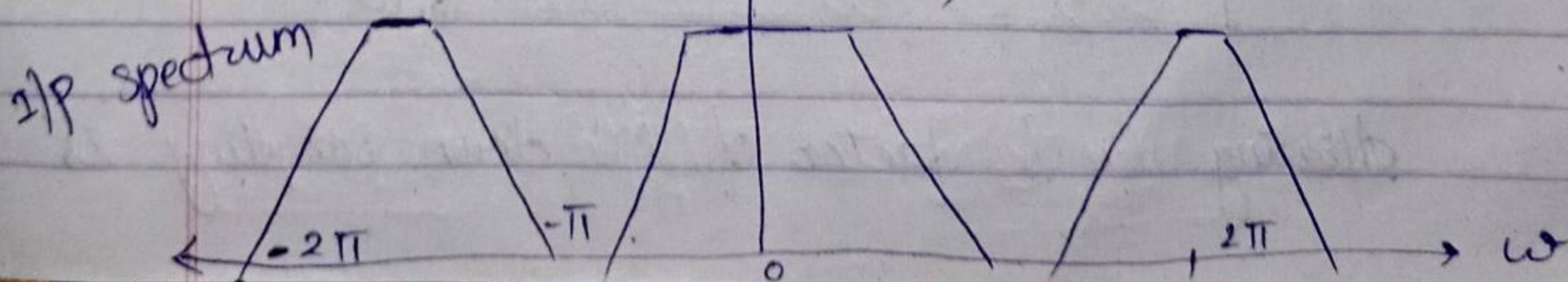
In similar manner, we can show that

$$X_u[z] = X(z^L)$$

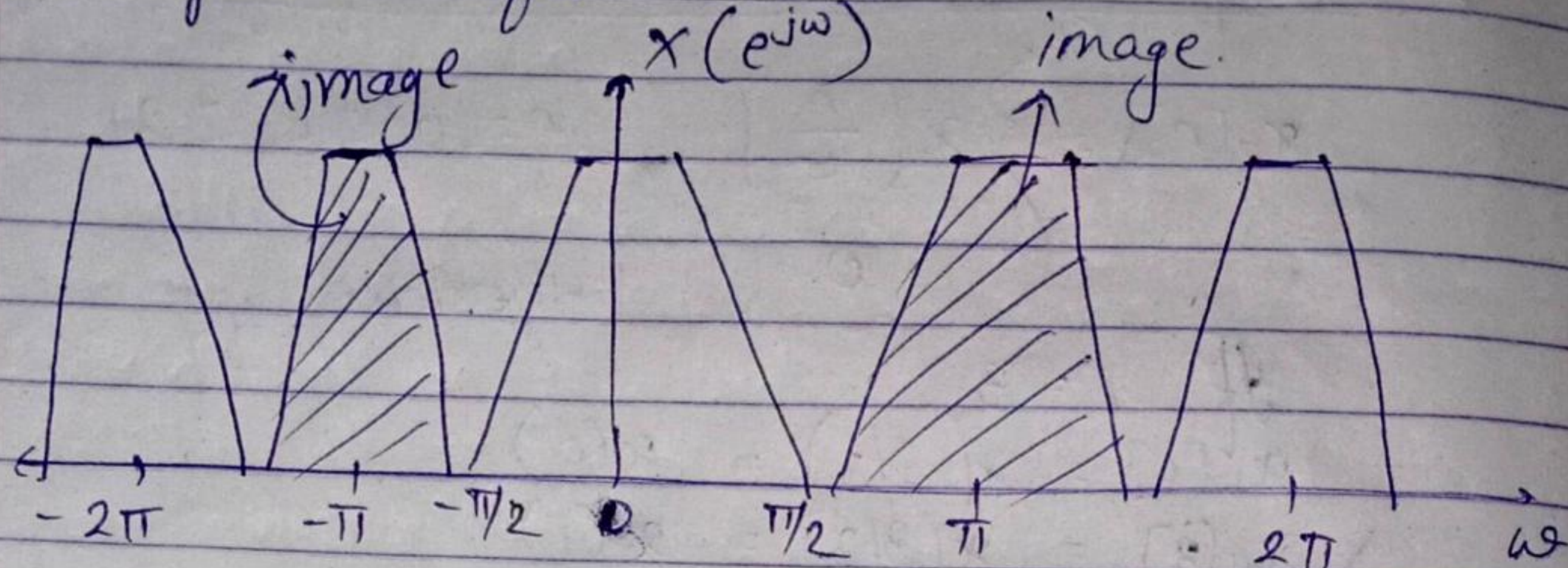
For $z = e^{j\omega}$ the eqⁿ becomes,

$$X_u(e^{j\omega}) = X(e^{j\omega L})$$

DTFT of $X(e^{j\omega})$ is $X(e^{j\omega})$ as follows,



o/p spectrum for $L=2$



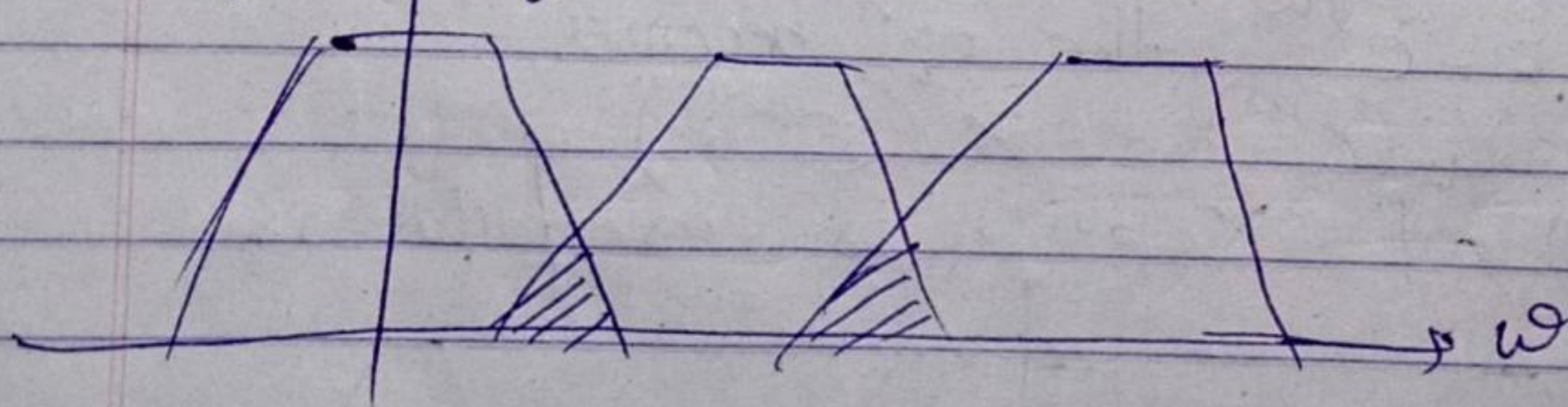
Fourier transform gets compressed by factor of 2.
($L-1$) no. of replicas.

- 1) The factor of 2 sampling rate expansion thus lead to two fold repetition of $X(e^{jω})$.
- 2) This indicates that Fourier transform is compressed by factor of 2.
- 3) This process is called as imaging bcoz we get additional images of I/P spectrum.
- 4) In case of sampling rate ' L ' we get ' $(L-1)$ ' additional images of I/P spectrum.

* Frequency domain representation of down sampler

- Original shape of the spectrum is lost when it is down sampled.
- Aliasing takes place.

$X(e^{jω/2})$ for $M=2$.



Aliasing due to factor of ' M ' down sampling is

absent iff the signal $x(n)$ is band limited to $\pm \frac{\pi}{M}$.

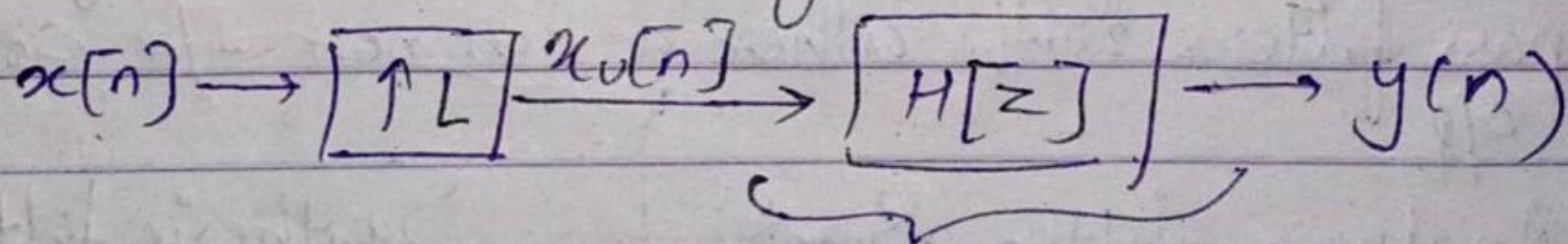
* Multirate structures for sampling rate conversion

- The process of reducing the sampling rate of a sequence is known as decimation & multirate structure is known as decimator.
- The process of increasing the sampling rate of the sequence is known as interpolation & multirate structure is known as interpolator.

Basic structure,

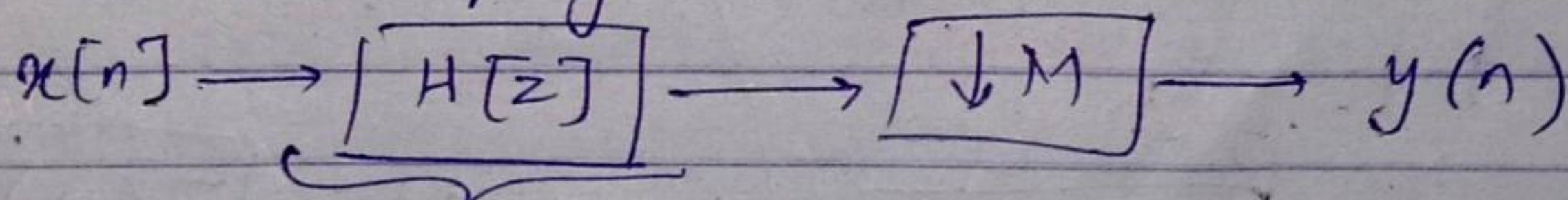
- up sampling by integer factor 'L' causes periodic repetition of basic spectrum. Imaging effect occurs.
- To remove $(L-1)$ extra unwanted images in the spectra of upsampled signal $x_u[n]$, interpolation filter is used. It is low pass filter.

The structure is as follows,



~~Low pass~~ filter interpolator
 $H(z)$ is having cutoff freqⁿ = π/L .

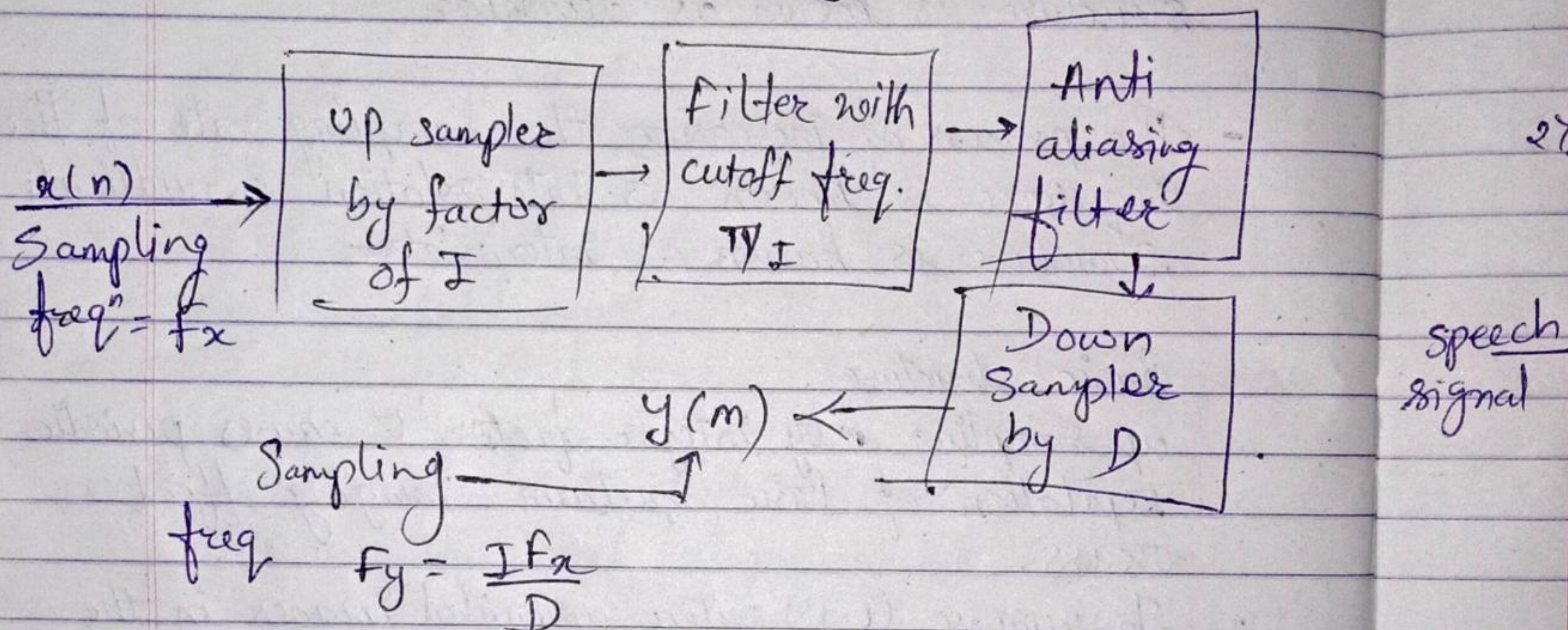
- For down sampling,



decimator filter
cutoff freqⁿ = π/M

★ Sampling rate conversion by factor of I/D .

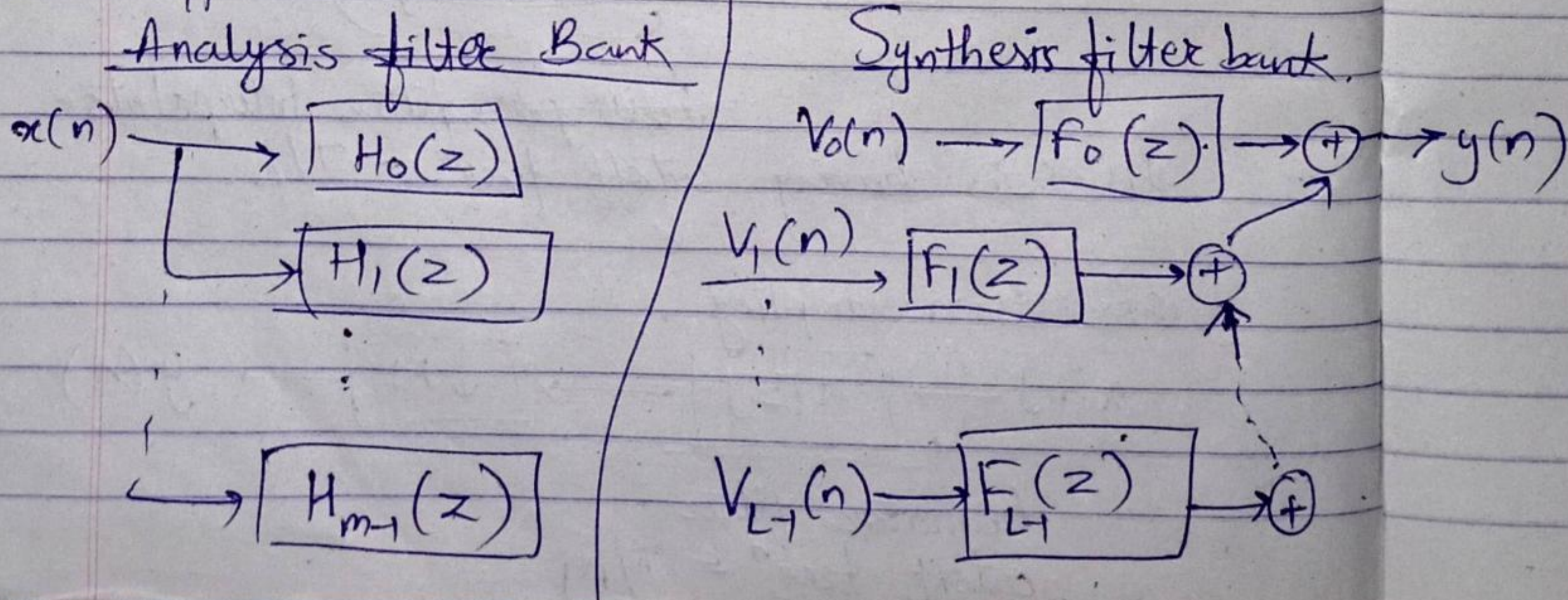
- both interpolation & decimation process is carried out.
- Usually interpolation is done first bcoz:
 ↳ Spectral characteristics of original signal $x(n)$ will be preserved.
 ↳ This allows us to combine the interpolator & decimator filter into single filter.



★ Applications of multirate DSP.

↳ Digital filter Bank.

Digital filter bank is set of digital band pass filters with either a common I/p or summed o/p.

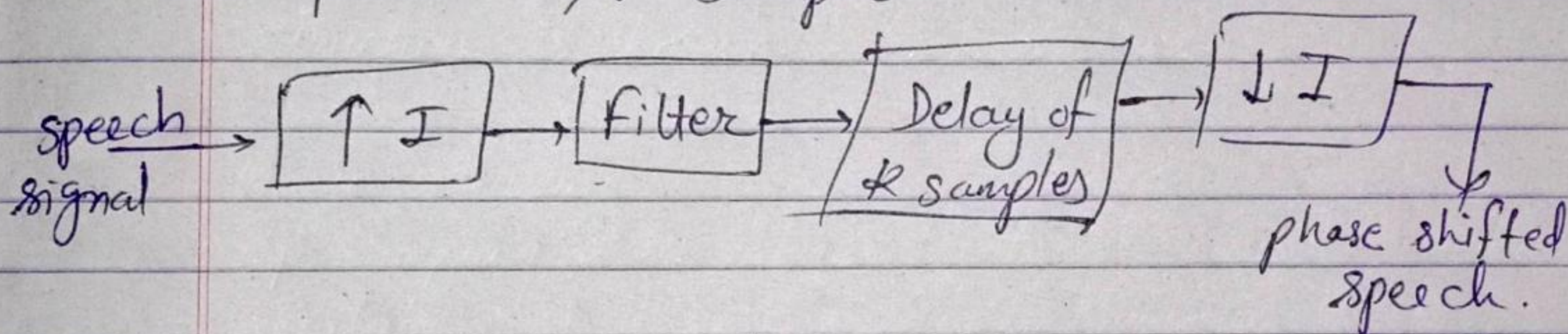


* Application of filter banks.

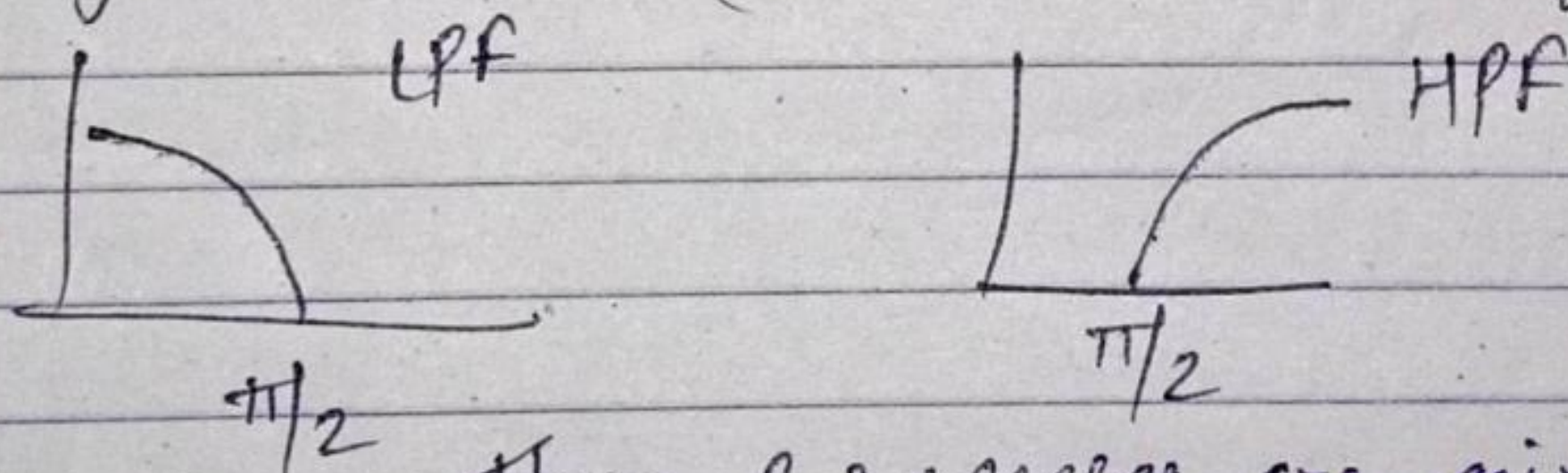
17 Spectrum Analyzer: In case of spectrum analyzer it is desirable to separate a signal into a set of sub band signals. Usually non overlapping portions of original frequency bands.

In other Application it may be necessary to combine many such sub band signals into a single composite signal occupying the whole nyquist range.

27 Design of phase shifter
phase K/I samples.

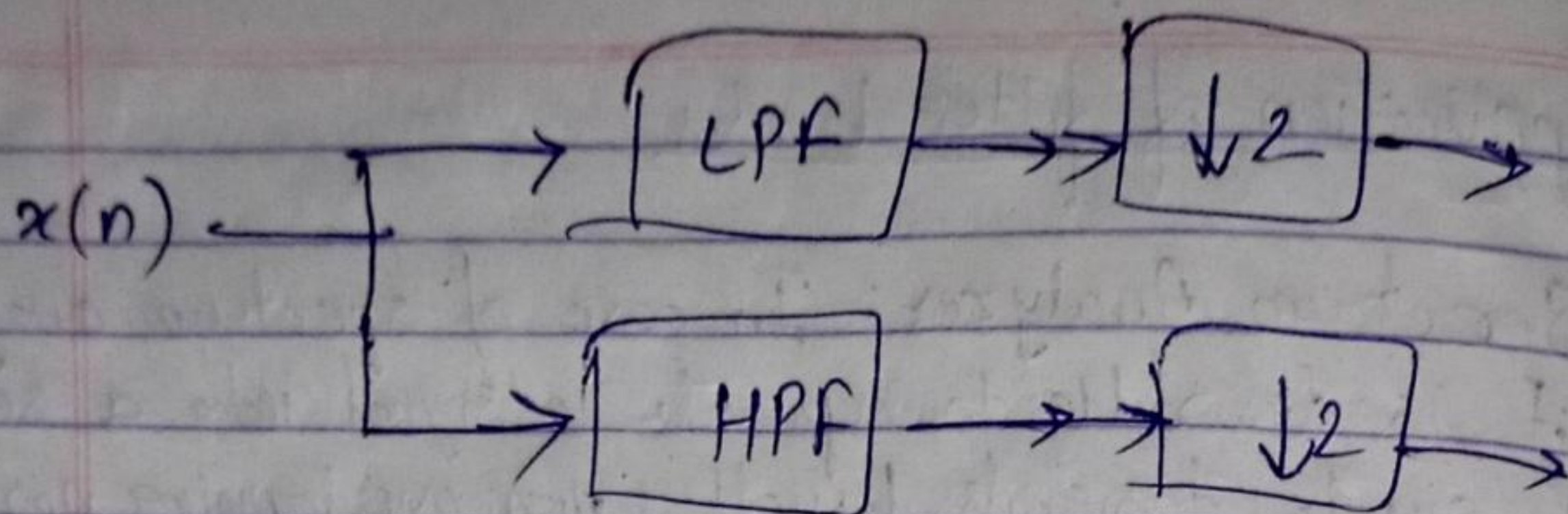


Design of QMF (Quadrature mirror filter)



These 2 responses are mirror images of each other about freq. $\pi/2$. These are called QMF.

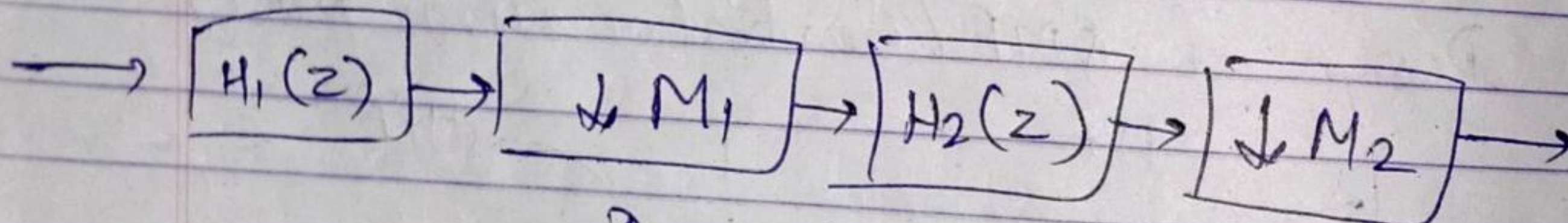
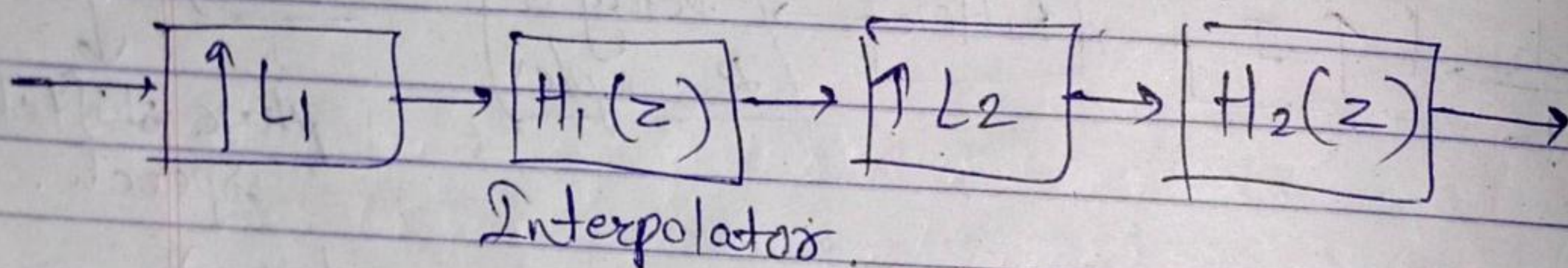
QMF are in phase quadrature. One filter is LPF with cutoff freq. $\pi/2$, & other is HPF with cutoff freq $\pi/2$. Frequency responses of 2 filters are mirror images of each other about $\pi/2$ Hence these filters are termed as QMF. Each filter has BW of $\pi/2$. Hence o/p of each filter can be safely down sampled by 2.



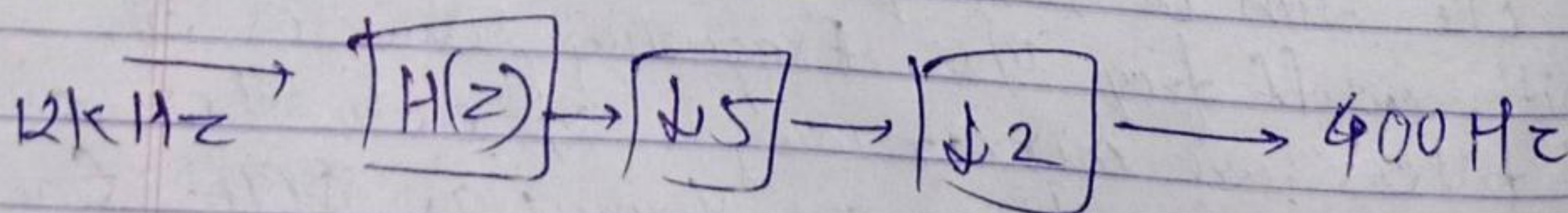
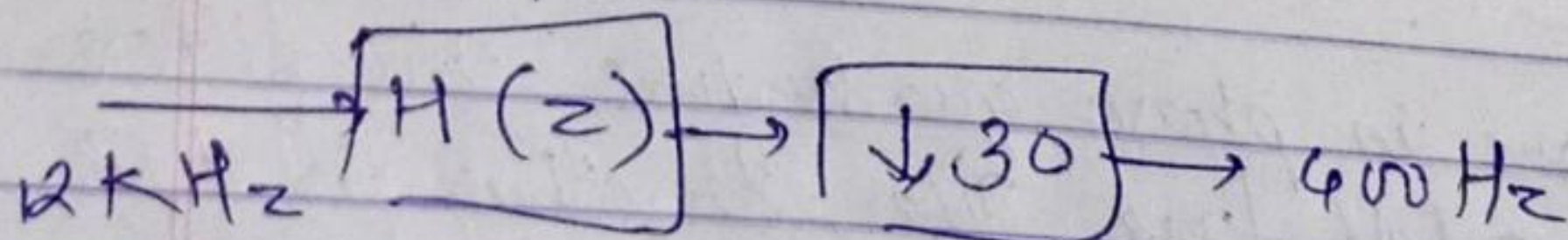
- * Image Resizing:
to reduce size of image $\rightarrow$ decimation
to increase size of image $\rightarrow$ interpolation.

* Multistage design of decimator & interpolator.
 $L = L_1 \times L_2$.

- * 2 stage implementation of sampling rate alteration device.



Decimator
12 K to 400 Hz
desired decimation factor is.
$$\frac{12 \text{ K}}{400} = 30$$



Computational complexity = No. of multiplications required per second.

$$R_m; \text{FIR} = N \times F_T$$

N = length of filter
 $F_T = \text{IP Sampling rate}$

Q7] A signal with sampling freq. of 3072 kHz is to be decimated by a factor of 3 to get new sampling freq. of 48 kHz. The signal band of interest extends from 0 to 20 kHz. The passband deviation is 0.01 dB & stopband attenuation is 80 dB. Design a 3 stage filter

→ We have to design a suitable anti aliasing filter

The decimation factor required is $\frac{3072K}{48K} = \underline{64}$

$= 8 \times 4 \times 2$

passband edge freq. of the stage is 24 kHz to preserve the band of interest.

Stop band edge frequency must be calculated. Let decimation factor is 8.

I/p Sampling freq. $f_1 = 3072 \text{ KHz}$

down sampling factor = 8

O/p sampling freq. $= \frac{3072}{8} = \underline{384 \text{ KHz}}$

When o/p sampling freq. = 384 KHz then the band of frequencies is upto 192 KHz (Nyquist freq.) is represented. $\hookrightarrow f_s/2 = 384/2$

The cutoff freq. for the filter is 192 KHz

The passband edge is 24 KHz.

stopband edge frequency at i^{th} stage (f_{si})

$$f_{si} = f_i - \frac{f_s}{2 \times M}$$

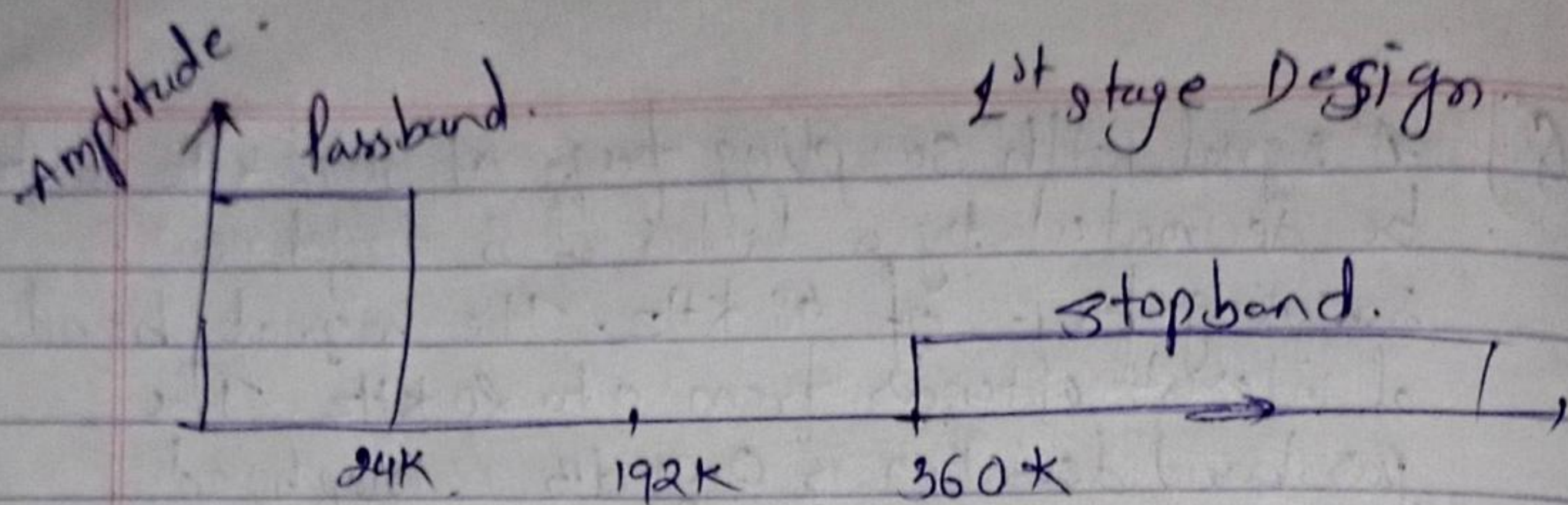
M = overall decimation factor

f_{si} = sampling freq. at i^{th} stage.

The stop band edge freq.

$$f_{s1} = f_1 - \frac{f_s}{2 \times M}$$

$$f_{s1} = 384 - \frac{3072}{2 \times 64} = \underline{360 \text{ KHz}}$$



Second stage Design.

decimation factor = 4

I/P sampling freq. = 384 kHz.

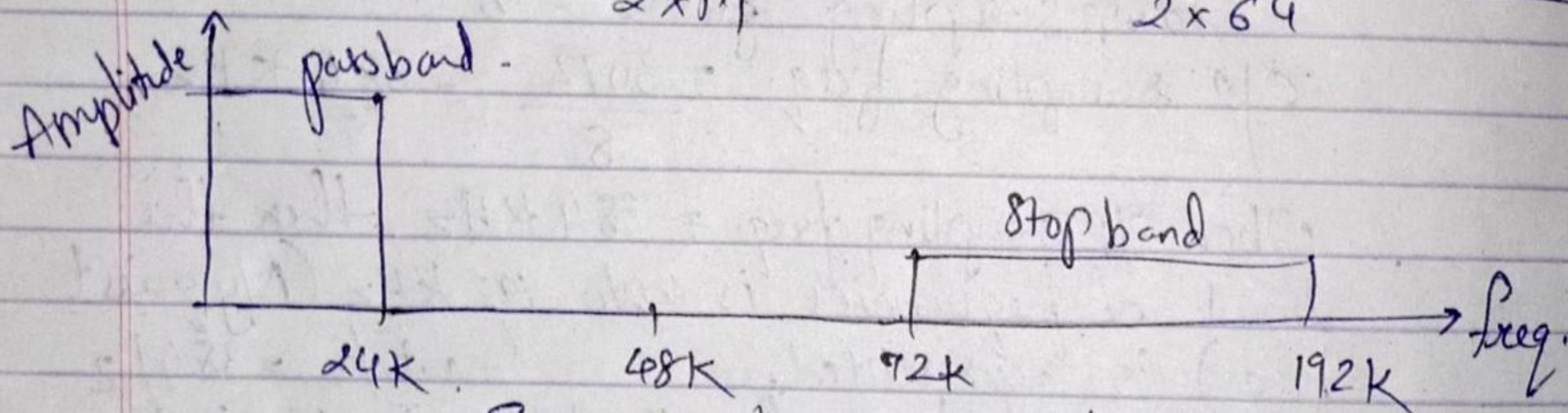
O/P sampling freq. with decimation factor 4
 $= 384\text{K} / 4 = 96\text{KHz}$.

Nyquist sampling freq. = $96 / 2 = 48\text{KHz}$.

cutoff freq. = 96 kHz.

pass band edge = 24 kHz.

$$f_{s2} = f_g = \frac{f_s}{2 \times M} = 96 - \frac{3072}{2 \times 64} = \underline{\underline{72\text{KHz}}}$$



Second stage specifications.

Third stage Design.

decimation factor = 2

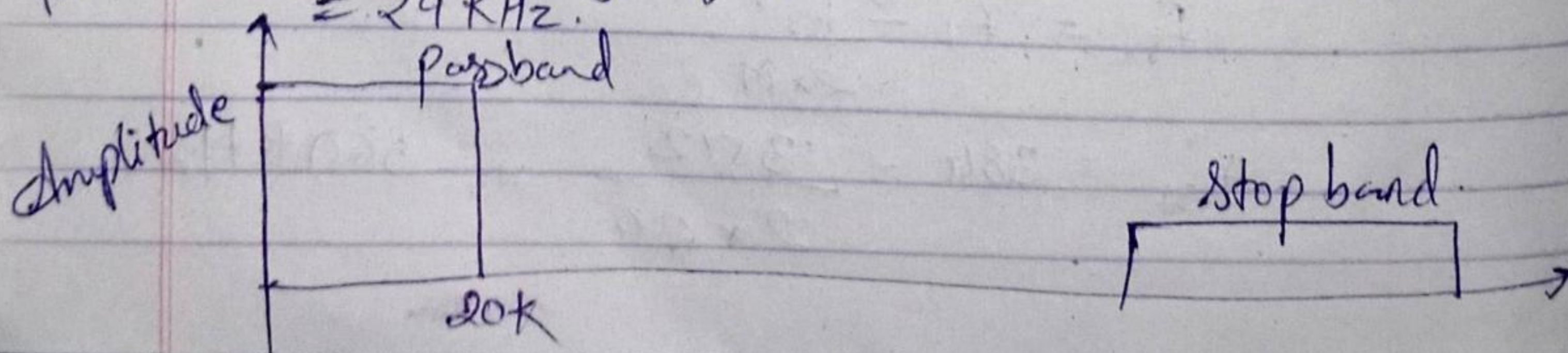
I/P sampling freq. = 96 kHz.

O/P sampling freq. = $96 / 2 = 48\text{KHz}$.

Nyquist sampling freq. = $48 / 2 = 24\text{KHz}$.

pass band edge freq. = 20 kHz.

stop band edge freq. must be calculated.
 $= 24\text{KHz}$.



Number of multiplications required per second

$$N_1 f_1 + N_2 f_2 + N_3 f_3 = (8 \times 384) + (4 \times 96) + (2 \times 48) \\ = 3552 \text{ k}.$$

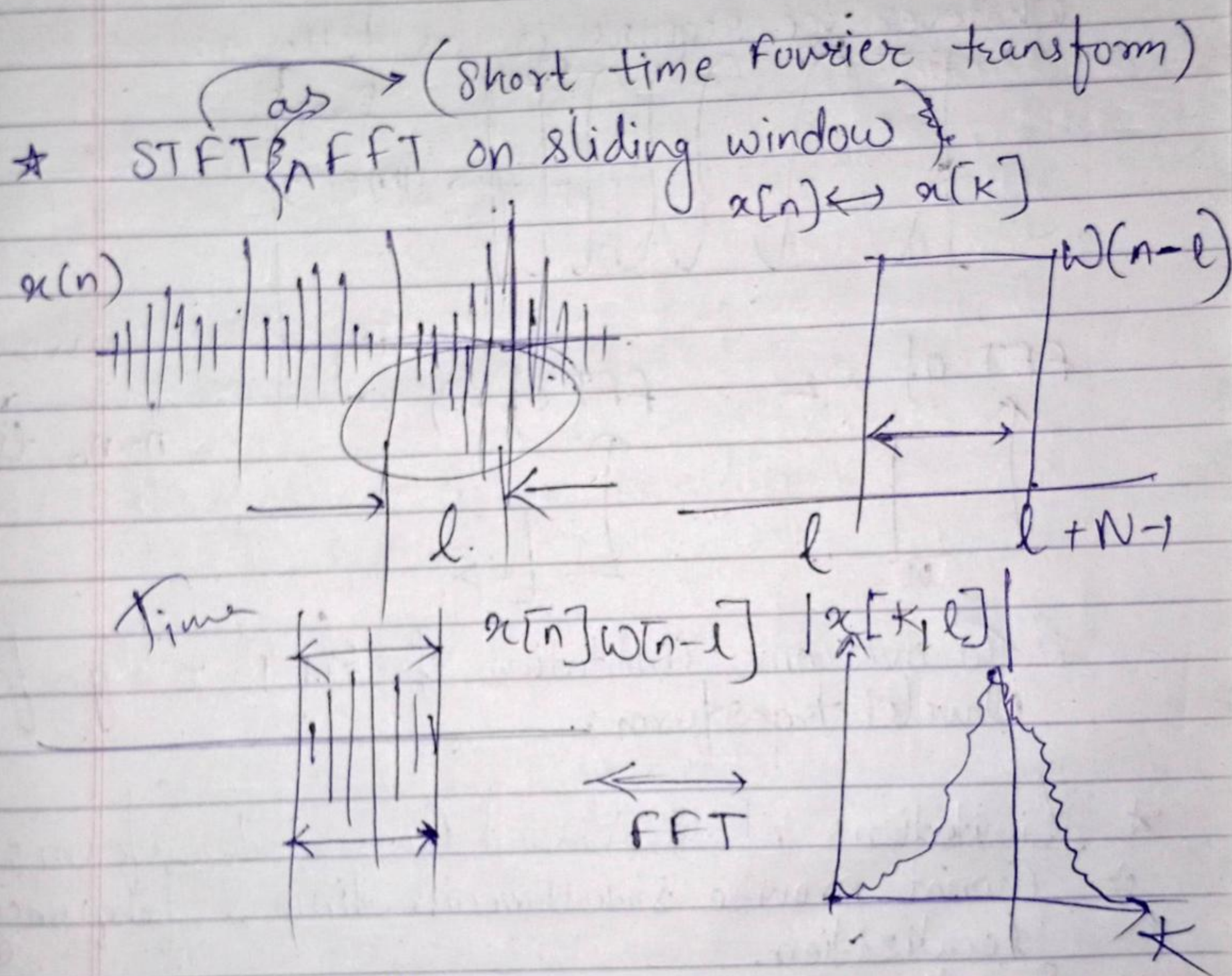
The pass band ripple gets distributed over 3 filters. Stop band attenuation is same for all stages:

This is because stop band of the first stage gets eliminated from the second stage.

Module 6.

Address the problem of representing signal in the form of instantaneous spectrum of the signal.

→ achieved by simple extension of DFT.



STFT is defined as,

$$x[k, l] = \text{DFT} \{ x[l]w[0], \dots, x[l+N-1]w[N-1] \}$$

$$\forall k = 0, \dots, N-1$$

where k = index denoting freq.

l = denotes time.

As $x[k, l]$ is FFT, it represents magnitude & phase of frequency.

$$f = \frac{k F_s}{K}$$

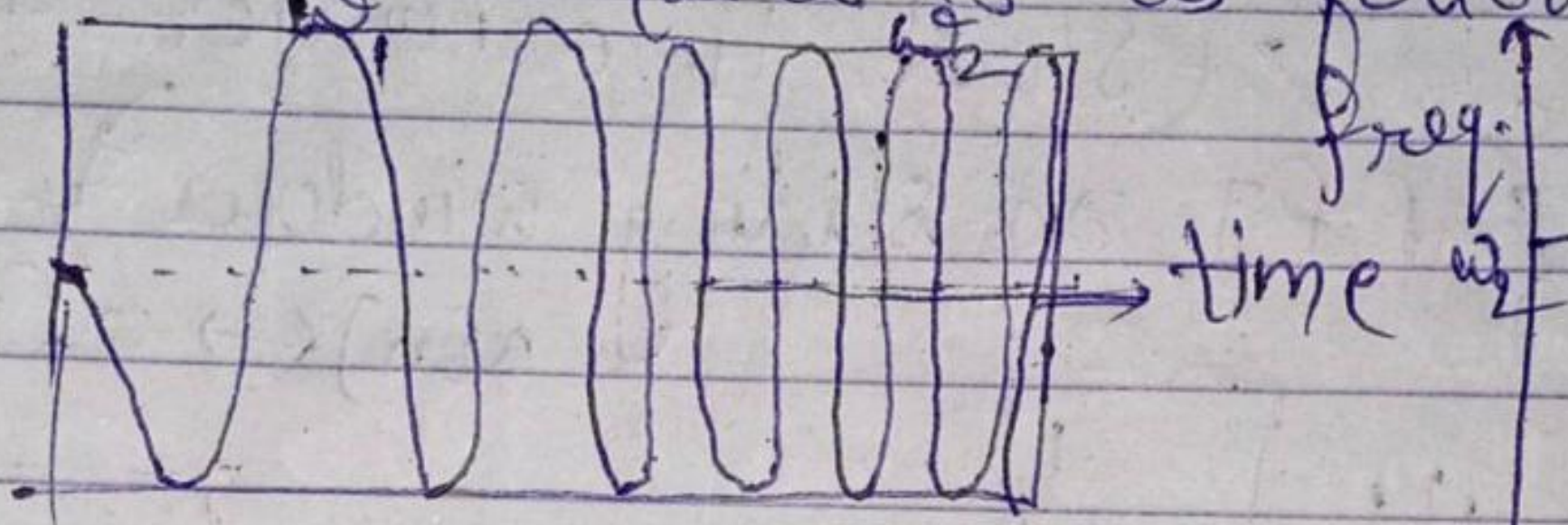
at time $t = lT_s$

$$T_s = \frac{1}{F_s} = \text{Sampling interval.}$$

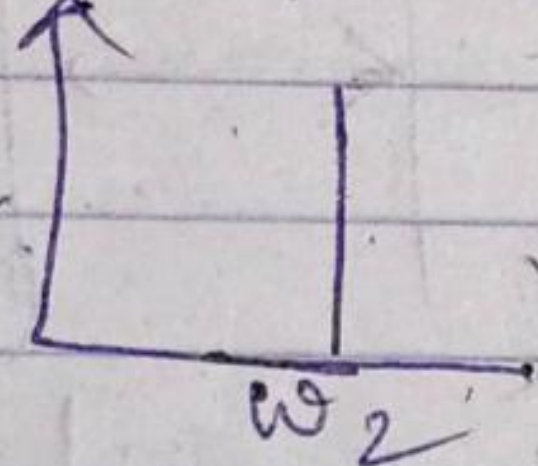
$$|x[k, l]|$$

related to time frequency

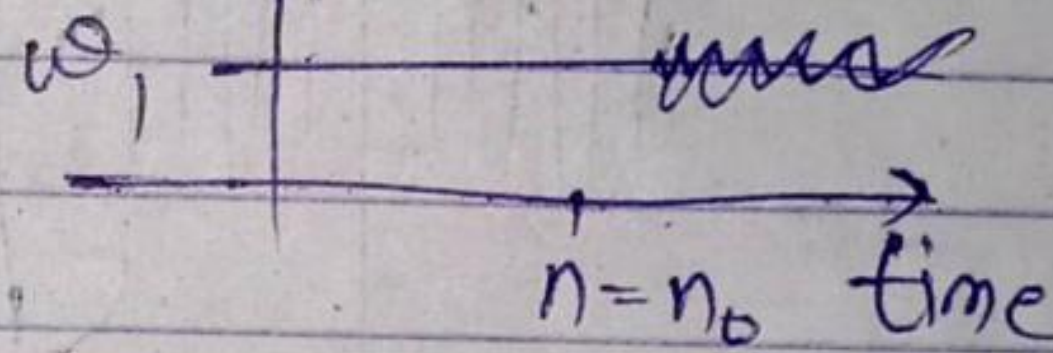
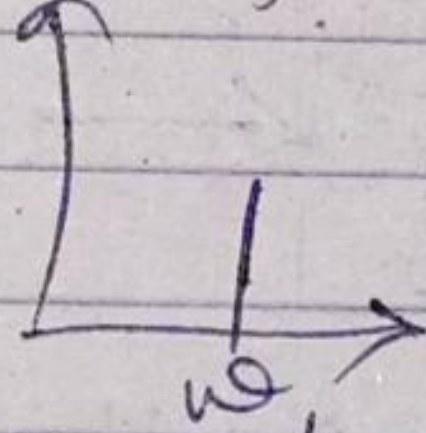
Whatever we require is as follows:



FFT of w_2



FFT of w_1



To overcome limitation of STFT, we go for Wavelet transform.

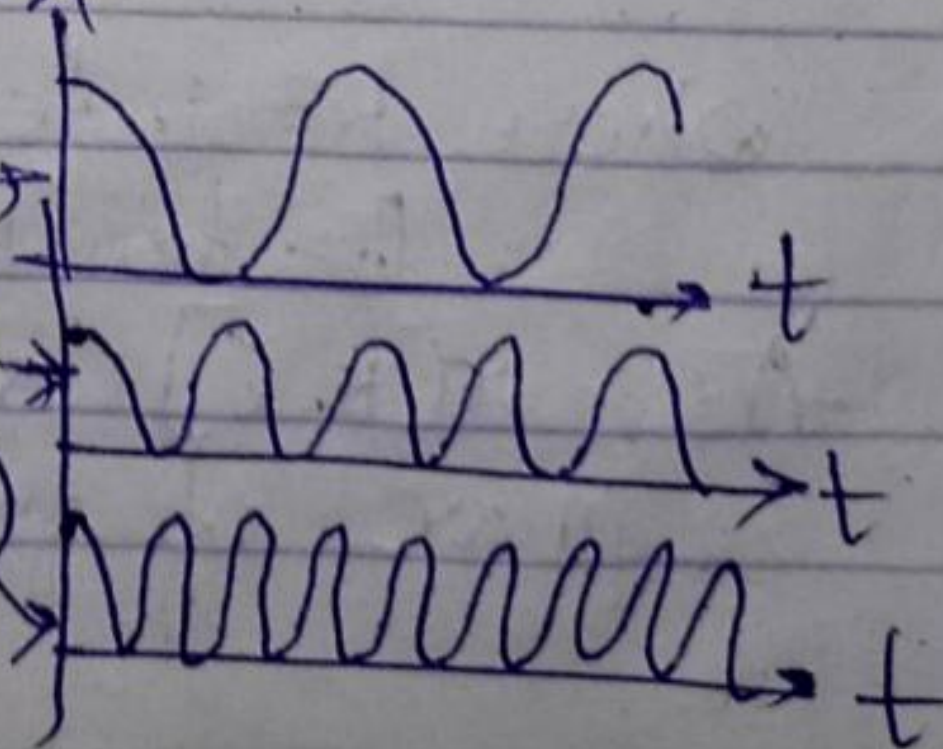
* Limitations of Normal Fourier transform:

- 1) Cannot provide simultaneous time & frequency localization.
- 2) Provides excellent localization in frequency domain but poor localization in time domain.
- 3) Not useful for analyzing time variant non stationary signals.

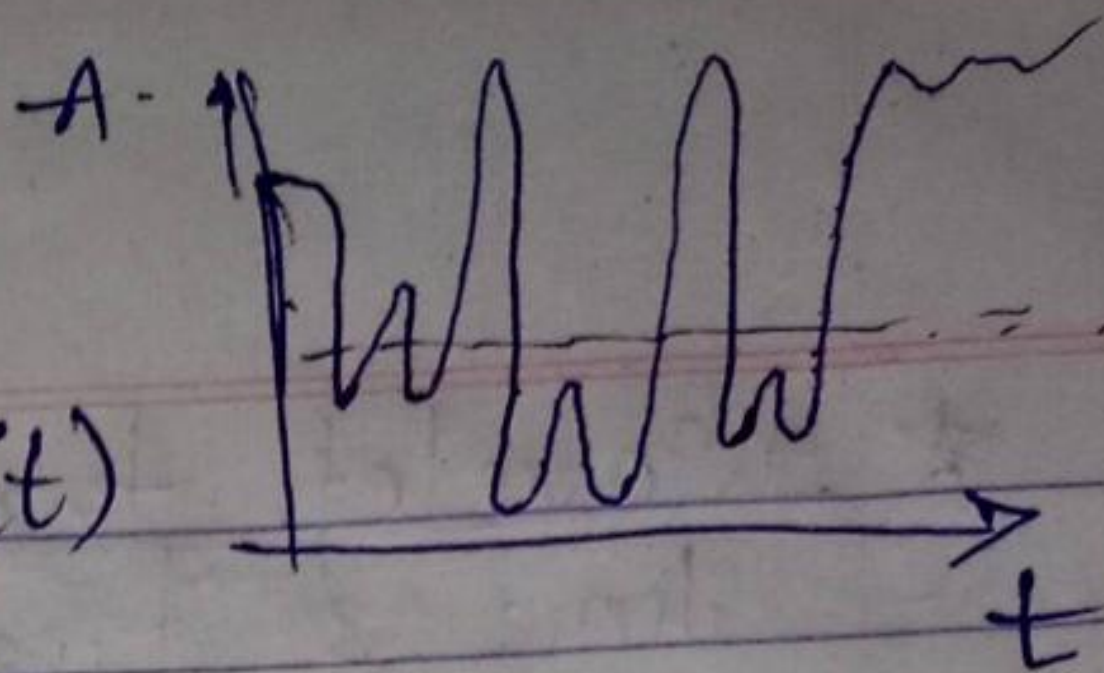
$$f_1(t) = \cos(2\pi \times 5t)$$

$$f_2(t) = \cos(2\pi \times 25t)$$

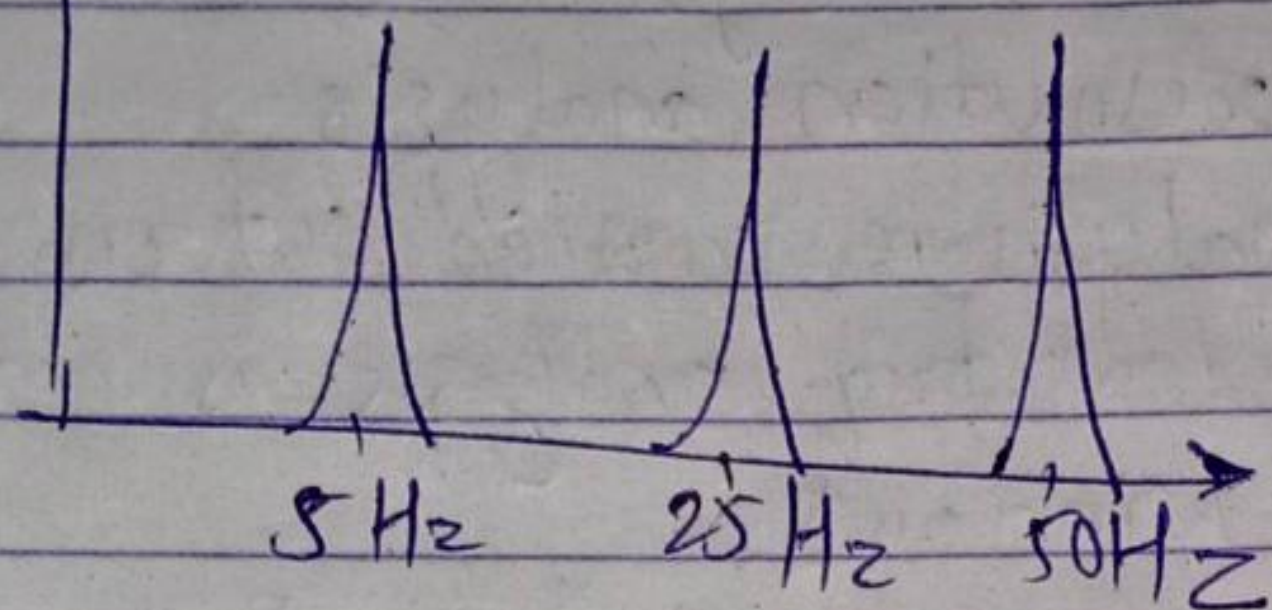
$$f_3(t) = \cos(2\pi \times 50t)$$



$$x(t) = f_1(t) + f_2(t) + f_3(t)$$



F.T.



$$x[n]w[n-l] \xleftrightarrow{\text{FFT}} X[k, l]$$

$$X[k, l] = \text{DFT} \{ x[l]w[0], \dots, x[l+N-1]w[N-1] \}$$

$$k = 0, \dots, N-1$$

k = index denoting freq.

l = denotes time.

Also called Gabor transform. (1940)

STFT defined as,

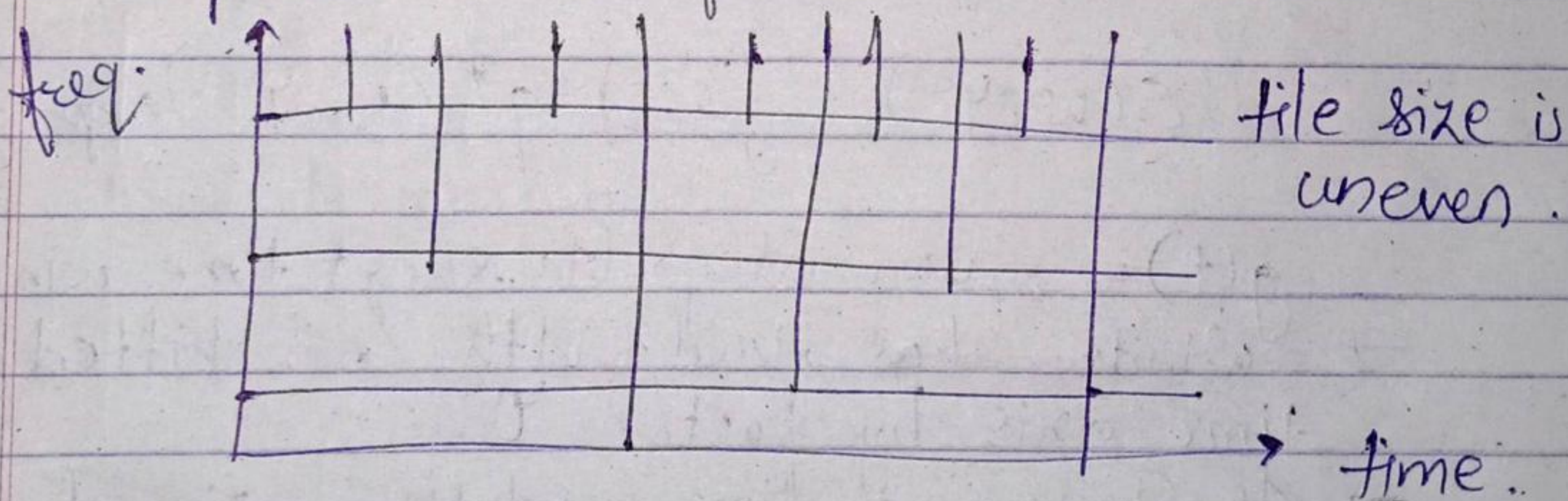
$$S(\omega, \tau) = \int f(t) g^*(t - \tau) \exp(-j\omega t) dt$$

- $g(t)$ = square integrable short time window.
- Window has fixed width & is shifted along time axis by factor τ .
- As Increase in time resolution, size of window decreases. & vice versa.

★ Wavelet transform [WT]

- time & freq. representation of signal simultaneously.
- We can resolve the signal simultaneously.
- provides multiresolution analysis.
- High freq. analysis is carried out using narrower window. & low freq. analysis is carried out using wider windows.
- When we use narrow window, time resolution is better & high frequencies are resolved in time domain.
- When we use wider window, freq. resolution is better & low frequencies are resolved in freq. domain.
- In joint time & freq. response representation, we can resolve low & high frequencies simultaneously.

~~Time~~ This is time & freq. tradeoff & represented as follows.



Relative Bandwidth.

$$Q = \frac{\text{BW of band}}{\text{centre freq. of band.}}$$

★ Basic Wavelet function → the basis function can be generated from a basic wavelet function. Basic Wavelet function is mother wavelet.

or kernel.

→ Child wavelet are generated by scaling mother wavelet.

The set of basis functions contains time domain windows of different lengths, so wavelet can resolve the high freq & low freq. signals simultaneously.

★ Wavelet transform.
low frequencies $\rightarrow$ "freq" domain
high frequencies $\rightarrow$ Time domain

The signal is represented as weighted sum of basis function.

The wavelet transform coefficient at scale 'a' & translation 'b' is given as

$$W(a,b) = \int_{-\infty}^{\infty} s(t) h_{a,b}^*(t) dt$$

Here

$$h_{a,b}(t) = \left(\frac{1}{a}\right)^{1/2} g\left(\frac{t-b}{a}\right)$$

The wavelet coefficient represents how well the signal $s(t)$ is similar to the scaled & translated mother wavelet.

* Restrictions on mother wavelet (kernel)

1) If $g(t)$ is mother wavelet, then,
2) $g(t)$ has finite energy.

$$\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$$

This condition is called as regularity condition.

2) Fourier transform of $g(t)$ is finite. FT of $g(t)$ must have zero value at zero frequency.
WT removes DC component of a signal.

* Types of wavelet

- 1) Continuous wavelet transform.
- 2) Discrete wavelet transform.
 - 1) Haar wavelet transform.
 - 2) Daubechies wavelet

Haar wavelet - continuous WT

Haar wavelet is given as,

$$\psi(t) = \begin{cases} 1, & 0 < t < 1/2 \\ -1, & 1/2 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

The scaling function is defined as,

$$\phi(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

The wavelet coefficients at scale 'a' & translation 'b' of a signal $s(t)$ can be written as,

$$w(a, b) = \int_{-\infty}^{\infty} s(t) h_{a, b}^*(t) dt.$$

$$h_{a, b}(t) = \frac{1}{\sqrt{a}} g\left[\frac{(t-b)}{a}\right]$$

$g(t)$ = mother wavelet.

$s(t)$ = weighted sum of basis function.

* Properties of Haar Wavelet.

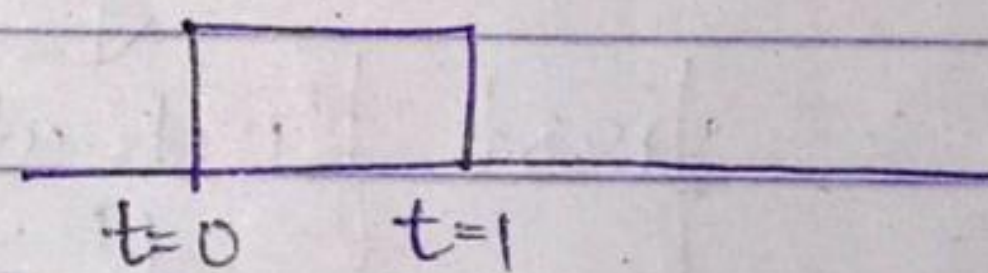
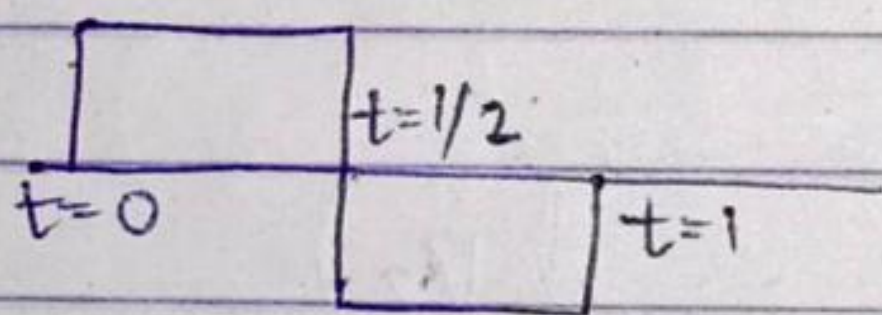
- 1) It is real & orthogonal.
- 2) It is very fast transform.
- 3) 2×2 Haar matrix is given as,

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Haar wavelet $\rightarrow$ sequence of rescaled square shaped functions which together forms a wavelet family or basis.

$\psi(t)$ mother wavelet.

$\phi(t)$ scaling function



- Haar transform decomposes a discrete signal into two sub signals of half its length.
- one sub signal is called as running average or trend.
- other subsequence is called as running difference or fluctuation.

- The first trend sub signal is

$$a' = (a_1, a_2, \dots, a_{N/2})$$

a' calculated by taking the average of the first pair of values of f .

$\frac{f_1 + f_2}{2}$ then multiplying it by $\sqrt{2}$

$$\therefore a' = \frac{f_1 + f_2}{\sqrt{2}}$$

$$f = (f_1, f_2, f_3, f_4, f_5, f_6) \quad \text{---}$$

$$a^2 = \frac{f_3 + f_4}{\sqrt{2}}$$

$$a_m = \frac{f_{2m-1} + f_{2m}}{\sqrt{2}}$$

$$\forall m = 1, 2, 3, \dots, N/2$$

$$\text{eg} \rightarrow f = (4, 6, 10, 12, 8, 6, 5, 5)$$

$$a_1 = a' = \frac{4+6}{\sqrt{2}} = 5\sqrt{2}$$

$$a' = 5\sqrt{2}$$

$$a_2 = 7\sqrt{2}$$

$$\therefore a' = (5\sqrt{2}, 11\sqrt{2}, 7\sqrt{2}, 5\sqrt{2})$$

$$a_3 = 11\sqrt{2}$$

$$a_4 = 5\sqrt{2}$$

Multiplication by $\sqrt{2}$ is needed to ensure that the Haar transform preserves energy of signal.

$$a_m = \frac{f_{2m-1} + f_{2m}}{\sqrt{2}}$$

first fluctuation

$$d' = (d_1, d_2, \dots, d_{N/2})$$

$$d_1 = \frac{f_1 - f_2}{\sqrt{2}}$$

$$d_m = \frac{f_{2m-1} - f_{2m}}{\sqrt{2}}$$

$$f = \{4, 6, 10, 12, 8, 6, 5, 5\}$$

$$d = \{-1, -1, 1, 0\}$$

$$d' = \{-\sqrt{2}, -\sqrt{2}, \sqrt{2}, 0\}$$

* Haar transform

The first level is mapping H_1 defined by

$$f \xrightarrow{H_1} (a', d')$$

$$(4, 6, 10, 12, 8, 6, 5, 5) \xrightarrow{H_1} (5\sqrt{2}, 11\sqrt{2}, 7\sqrt{2}, 5\sqrt{2} / -\sqrt{2}, -\sqrt{2}, \sqrt{2}, 0)$$

The mapping H_1 has inverse. (a'/d')

$$f = \left(\frac{a_1 + d_1}{\sqrt{2}}, \frac{a_1 - d_1}{\sqrt{2}}, \dots, \frac{a_{N/2} + d_{N/2}}{\sqrt{2}}, \frac{a_{N/2} - d_{N/2}}{\sqrt{2}} \right)$$

$$f_1 = \frac{(a_1 + d_1)}{\sqrt{2}}, f_2 = \frac{a_1 - d_1}{\sqrt{2}}, f_3 = \frac{a_2 + d_2}{\sqrt{2}}, f_4 = \frac{a_2 - d_2}{\sqrt{2}}$$

Conservation & compaction of energy.

- 1) It conserves energies of signal
- 2) It performs compaction of the energy of signals.

* Conservation of energy.

Energy of signal f_2

$$E_f = f_1^2 + f_2^2 + \dots + f_N^2$$

$$f = \{4, 6, 10, 12, 8, 6, 5, 5\}$$

$$E_f = 4^2 + 6^2 + \dots = 446$$

1st level of Haar transform.

$$(a'/d') = (5\sqrt{2}, 11\sqrt{2}, 7\sqrt{2}, 5\sqrt{2} / -\sqrt{2}, -\sqrt{2}, \sqrt{2}, 0)$$

$$E(a'/d') = (25 \times 2 + 121 \times 2 + \dots) = 446$$

* Daubachis Wavelet.

Basis function: family of orthonormal, compactly supported scaling & wavelet functions that has maximum regularity for given length of mirror filter wavelet basis of different lengths 4, 6, 8, etc.

Basis for D_2 containing 4 filter coefficients are given as set of LPF coefficients, f_n (wavelet fn.) HPF coefficients (scaling fn.) $g(n)$

$$p(n) = \left(\frac{1+\sqrt{3}}{4\sqrt{2}}, \frac{3+\sqrt{3}}{4\sqrt{2}}, \frac{3-\sqrt{3}}{4\sqrt{2}}, \frac{1-\sqrt{3}}{4\sqrt{2}} \right) \\ = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$$

$$g(n) = \left(\frac{1-\sqrt{2}}{4\sqrt{2}}, \frac{-3-\sqrt{3}}{4\sqrt{2}}, \frac{3+\sqrt{3}}{4\sqrt{2}}, \frac{-1+\sqrt{3}}{4\sqrt{2}} \right) \\ = (\alpha_3, \alpha_2, \alpha_1, -\alpha_0)$$

($1/\sqrt{2}$ = Normalization factor.)

* Discrete Wavelet transform.

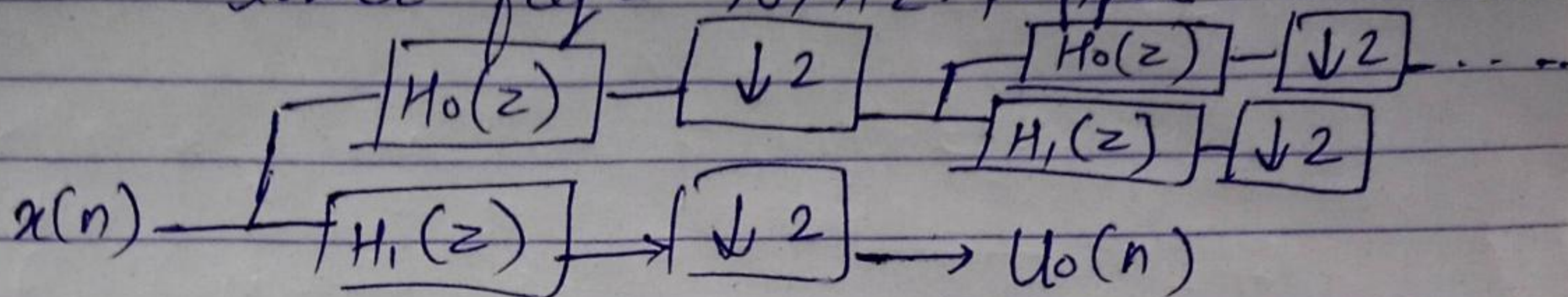
The fn. of octave band analysis filter bank with down sampling, also called as binary tree.

I/P sequence into subsequences at the o/p of down samples.

* octave band = a band is said to be octave in width upper band frequency is twice the lower band frequency.

eg $\rightarrow$ centre freq = 1 kHz.

lower freq = 707 Hz, upper = 1.414 kHz



$\leftarrow$ level 1 $\longleftrightarrow$ level 2 $\rightarrow$
 $H_1(z) \rightarrow$ Highpass analysis filter.
 $H_0(z) \rightarrow$ Lowpass analysis filter.

First stage.

Analysis filter has pass band of width $(0, \pi/2)$
 Highpass $[\pi/2, \pi]$

Second level LPF $\rightarrow [0, \pi/4]$

Second level HPF $\rightarrow [\pi/4, \pi/2]$

The process of generating the set of o/p sequences $U_k[n]$, $0 \leq k \leq L$ from finite length I/p sequence $x[n]$ using the analysis filter of level octave band perfect reconstruction QMF bank is known as discrete wavelet transform (DWT).

The basic impulse response of HPF, given by inverse Z transform of $H_1(z)$ is called as mother wavelet.

* Impulse response of LPF is called as scaling fn. or child wavelet.